HEWLETT-PACKARD

Step-by-Step Examples for Your HP-28C

Calculus

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HEWLETT PACKARD

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Welcome...

... to the HP-28C Step-by-Step Booklets. These books are designed to help you get the most from your HP-28C calculator.

This booklet, *Calculus*, provides examples and techniques for solving problems on your HP-28C. A variety of function operations and differential and integral calculus problems are designed to familiarize you with the many functions built into your HP-28C.

Before you try the examples in this book, you should be familiar with certain concepts from the owner's documentation:

- The basics of your calculator – how to move from menu to menu, how to exit graphics and edit modes, and how to use the menu to assign values to, and solve for, user variables.
- Entering numbers and algebraic expressions into the calculator.

Please review the section "How to Use This Booklet." It contains important information on the examples in this booklet.

For more information about the topics in the *Calculus* booklet, refer to a basic textbook on the subject. Many references are available in university libraries and in technical and college bookstores. The examples in the booklet demonstrate approaches to solving certain problems, but they do not cover the many ways to approach solutions to mathematical problems.

Our thanks to Ross Greenley of Oregon State University for developing the problems in this book.

Contents

How To Use This Booklet

Please take a moment to familiarize yourself with the formats used in this booklet.

Keys and Menu Selection: A box represents a key on the calculator keyboard.

[ENTER]
[1/x]
[STO]

[ARRAY]
[PLOT]
[ALGEBRA]

In many cases, a box represents a shifted key on the HP-28C. In the example problems, the shift key is NOT explicitly shown (for example, [ARRAY] requires the press of the shift key, followed by the ARRAY key, found above the "A" on the left keyboard).

The "inverse" highlight represents a menu label:

≣DRAW≣ (found in the [PLOT] menu)
≣ISOL≣ (found in the [ALGEBRA] menu)
≣ABCD≣ (a user-created name, found in the [USER] menu)

Menus typically include more menu labels than can be displayed above the six redefinable menu keys. Press [NEXT] and [PREV] to roll through the menu options. For simplicity, [NEXT] and [PREV] are NOT shown in the examples.

Solving for a user variable within ≣SOLVR≣ is initiated by the shift key, followed by the appropriate user-defined menu key:

[ ] ≣ABCD≣.

The keys above indicate the shift key, followed by the user-defined key labeled "ABCD". Pressing these keys initiates the Solver function to seek a solution for "ABCD" in a specified equation.

The symbol [<>] indicates the cursor-menu key.

Interactive Plots and the Graphics Cursor: Coordinate values you obtain from plots using the [INS] and [DEL] digitizing keys may differ from those shown, due to small differences in the positions of the graphics cursor. The values you obtain should be satisfactory for the Solver root-finding that follows.

Display Formats and Numeric Input: Negative numbers, displayed as

```
-5
-12345.678
[[-1,-2,-3 [ -4,-5,-6 [ ...
```

are created using the [CHS] key.

```
5 [CHS]
12345.678 [CHS]
[[1 [CHS] ,2 [CHS] , ...
```

The examples in this book typically specify a display format for the number of decimal places. If your display is set such that numeric displays do not match exactly, you can modify your display format with the [MODE] menu and the ≣FIX≣ key within that menu (e.g. [MODE] 2 ≣FIX≣).

Function Operations

The primary goals of this section are to write user-defined functions and introduce the root finding, plotting, and calculus capabilities of the HP-28C. Problems include definition and assignment of the trigonometric co-functions in the USER menu, analysis of a cubic equation, and computation of the angle between two intersecting lines in a specific and general case.

Function Definition

This section demonstrates creation of simple user-defined functions. The use of functions of this type is basic to efficient use of the HP-28C.

Example: The HP-28C has three basic trigonometric functions built in – sine, cosine, and tangent. It is simple to add the remaining co-functions to the USER menu. Built-in functions of the HP-28C can be easily combined to create new functions. The use of programs and local variables permits the newly defined functions to be used in the same manner as the built-in functions.

The inverse of the sine is the cosecant.

```
CLEAR  <>
« → x '1÷SIN(x
ENTER
```

```
4:
3:
2:
1:    « → x '1/SIN(x)' »
```

Store the user-defined function.

```
'CSC  STO
```

```
4:
3:
2:
1:
```

The inverse of the cosine is the secant.

```
« → x '1÷COS(x
ENTER
```

```
4:
3:
2:
1:    « → x '1/COS(x)' »
```

Store the user-defined function.

```
'SEC  STO
```

```
4:
3:
2:
1:
```

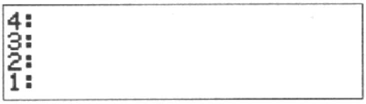
The inverse of the tangent is the cotangent.

```
« → x '1÷TAN(x
ENTER
```

```
4:
3:
2:
1:    « → x '1/TAN(x)' »
```

Store the user-defined function.

'COT [STO]



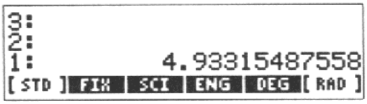
Example: Evaluate, in radians, COT(X) and $\mathrm{CSC}^2(X) - \mathrm{COT}^2(X)$, where X =.2.

First, store the value of X and select radians and standard display modes.

.2 [ENTER]
'X [STO]
[MODE] RAD STD

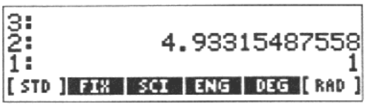
Now enter the expression for COT(X) and evaluate it.

'COT(X [ENTER]
[EVAL]



Enter the second expression and evaluate it.

'SQ(CSC(X))-SQ(COT(X))
[ENTER] [EVAL]



As expected, this identity returns the value 1.

Purge the user-defined functions and the variable X created in this section.

{CSC SEC COT X [ENTER] [PURGE]

Function Composition

This section demonstrates additional utility of user-defined functions. Arguments of the functions may be both numeric and symbolic.

Example: Form the compositions $F(G(x))$ and $G(F(x))$ given

$$F(x)=x^2+1 \text{ and } G(x)=5x+2.$$

Create F and G as user-defined functions.

First, create F.

[CLEAR] [< >]
« → X 'X^2+1 [ENTER]

```
4:
3:
2:
1:        « → x 'x^2+1' »
```

Store in the variable F.

'F [STO]

```
4:
3:
2:
1:
```

Now create G.

« → X '5 × X+2 [ENTER]

```
4:
3:
2:
1:        « → x '5*x+2' »
```

Store in the variable G.

'G [STO]

```
4:
3:
2:
1:
```

To form the composition $G(F(x))$, enter F as an argument of G.

'G(F(X [ENTER]

```
4:
3:
2:
1:              'G(F(X))'
```

Evaluate the composite function.

[EVAL]

```
4:
3:
2:
1:         '5*(X^2+1)+2'
```

This expression can be simplified using EXPAN and COLCT.

[ALGEBRA] ≣ EXPAN ≣

```
3:
2:
1:          '5*X^2+5*1+2'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

≣ COLCT ≣

```
3:
2:
1:               '7+5*X^2'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Repeat the process using G as an argument of F.

'F(G(X [ENTER]

```
3:
2:               '7+5*X^2'
1:               'F(G(X))'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Evaluate the composite function.

[EVAL]

```
3:
2:               '7+5*X^2'
1:          '(5*X+2)^2+1'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Simplify the expression.

≣ EXPAN ≣

```
2:               '7+5*X^2'
1:  '(5*X)^2+2*(5*X)*2+2
    ^2+1'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

≣ EXPAN ≣

```
2:               '7+5*X^2'
1:  '5*X*(5*X)+2*(5*X)*2
    +2*2+1'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

≣ COLCT ≣

```
3:
2:               '7+5*X^2'
1:        '5+25*X^2+20*X'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Purge the variables created in this problem section.

{F G [ENTER] [PURGE]

Function Analysis

The ability to locate extreme values and other key features of functions is critical to the solution of many problems in science and engineering. This section demonstrates the use of calculus to locate such features.

Example: Locate the roots, local maximum, mimimum, and inflection points of

$$F(x) = x^3 + 6x^2 + 11x + 6 .$$

Enter and name the given function.

[CLEAR] [<>]
'X^3+6 × X^2+11 × X+6
[ENTER]

```
4:
3:
2:
1:      'X^3+6*X^2+11*X+6'
```

'FN [STO]

```
4:
3:
2:
1:
```

Recall the function, enter the PLOT menu, and store it for plotting.

[USER] [FN]

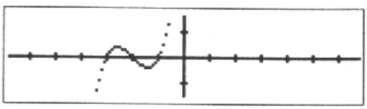
```
3:
2:
1:      'X^3+6*X^2+11*X+6'
 FN
```

[PLOT] [STEQ]

```
3:
2:
1:
STEQ RCEQ PMIN PMAX INDEP DRAW
```

Clear the plot parameters and plot the function.

'PPAR [PURGE]
[DRAW]



Digitize all of the roots.

< ... < INS
< ... < INS
< ... < INS
ATTN

```
3:                 (-.9,0)
2:                (-1.9,0)
1:                (-3.1,0)
STEQ RCEQ PMIN PMAX INDEP DRAW
```

Note: Differences from the displayed results may appear due to different digitizing locations.

Now enter the SOLV menu and compute the three roots.

SOLV SOLVR

```
3:                 (-.9,0)
2:                (-1.9,0)
1:                (-3.1,0)
 X  EXPR=
```

Enter a guess from the stack and compute the root.

X □ X

```
X: -3
Zero
1:                      -3
 X  EXPR=
```

After obtaining the exact root, make note of it and prepare to locate the next root. Discard the first root. Then repeat the process for the other two roots.

DROP
X □ X

```
X: -2
Zero
1:                      -2
 X  EXPR=
```

Compute the last root.

DROP
X □ X

```
X: -1
Zero
1:                      -1
 X  EXPR=
```

With the three roots located, find the extrema. The extrema are located by finding the roots of the first derivative.

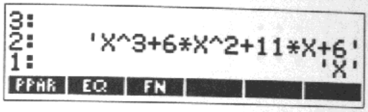
Recall the function.

CLEAR USER FN

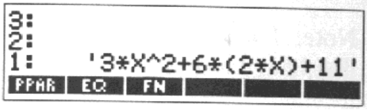
```
3:
2:
1:    'X^3+6*X^2+11*X+6'
 X  PPAR  EQ  FN
```

Purge the current value of X and differentiate with respect to X.

'X [ENTER] [ENTER]
[PURGE]

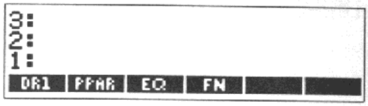


[d/dx]



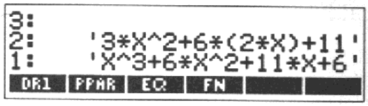
Store the first derivative.

'DR1 [STO]

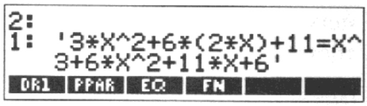


Plot the function and its first derivative.

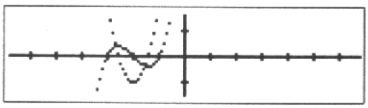
[DR1]
[FN]



[=] [ENTER]



[PLOT] [STEQ]
[DRAW]



Observe that the derivative is positive in regions where the function is increasing and negative in regions where it is decreasing.

Digitize both roots of the derivative.

[<] . . . [<] [INS]
[<] . . . [<] [INS]
[ATTN]

Note: Differences from the displayed results may appear due to differences in digitizing locations.

Recall the derivative and enter SOLVR to pinpoint the roots as was done previously. The computed values may differ slightly depending on the seed provided as an input to the Solver.

[USER] [DR1]
[SOLV] [STEQ]
[SOLVR]

```
3:
2:                (-1.4,0)
1:                (-2.6,0)
  X   EXPR=
```

[X]
[ ] [X]

```
X: -2.57735026917
Zero
1:        -2.57735026917
  X   EXPR=
```

This is one of the roots. Recall the function and evaluate to get the functional value.

[USER] [FN]
[EVAL]

```
3:                (-1.4,0)
2:        -2.57735026917
1:          .3849001794
  X   DR1  PPAR  EQ   FN
```

Now repeat the process for the other root. First discard the root and function value.

[DROP] [DROP]
[SOLV] [SOLVR]

```
3:
2:
1:                (-1.4,0)
  X   EXPR=
```

[X]
[ ] [X]

```
X: -1.42264973081
Zero
1:        -1.42264973081
  X   EXPR=
```

[USER] [FN]
[EVAL]

```
3:
2:        -1.42264973081
1:        -.38490017949
  X   DR1  PPAR  EQ   FN
```

The extreme values of the function have been located. Clear the stack and find the inflection point. The inflection point, located at the root of the second derivative, is the point or points at which the function changes concavity. That is, it changes from concave up to concave down. The second derivative of a cubic is linear and has only one root. Therefore a cubic has only one point of inflection.

Clear the value of X to obtain symbolic results.

[CLEAR]
'X [PURGE]

```
3:
2:
1:
DR1 PPAR EQ FN
```

Recall the first derivative.

≣ DR1 ≣
'X [ENTER]

```
3:
2:     '3*X^2+6*(2*X)+11'
1:                    'X'
DR1 PPAR EQ FN
```

Differentiate it with respect to X.

[d/dx]

```
3:
2:
1:            '3*(2*X)+12'
DR1 PPAR EQ FN
```

Store the second derivative.

'DR2 [STO]

```
3:
2:
1:
DR2 DR1 PPAR EQ FN
```

Plot the function and its second derivative. Observe the location of the root and how the function behaves at that point. It is coincidental that a function root is located at the point of inflection. It remains only to repeat the root finding procedure.

≣ DR2 ≣
≣ FN ≣

```
3:
2:            '3*(2*X)+12'
1:     'X^3+6*X^2+11*X+6'
DR2 DR1 PPAR EQ FN
```

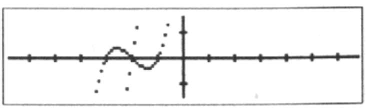
Set them equal for plotting.

[=] [ENTER]

```
2:
1:  '3*(2*X)+12=X^3+6*X^
    2+11*X+6'
DR2 DR1 PPAR EQ FN
```

Store and plot the equation.

[PLOT] ≣ STEQ ≣
≣ DRAW ≣

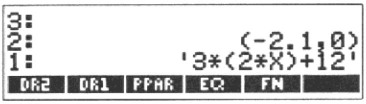


Digitize the root.

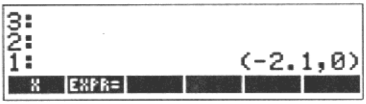
[<] . . . [<] [INS]
[ATTN]

Recall the second derivative and solve for the root.

[USER] DR2

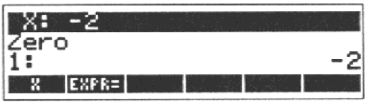


[SOLV] STEQ
SOLVR



Enter the digitized initial guess and solve for the root.

X
[■] X



This completes the analysis. We have found roots at $x = -1, -2, -3$, extrema at $x = -2.58, -1.42$ and an inflection point at $x = -2$.

Purge the user variables created in this section.

{FN X DR1 DR2 [ENTER] [PURGE]

Angle Between Two Lines

This section develops a user function to compute the angle of intersection of two lines. The slopes of the intersecting lines are supplied as arguments. The user function is used in the subsequent section in computing the angle of intersection of two general functions.

Example: Compute the angle between the lines

$$Y = 3x + 1 \text{ and } Y = -2x + 5 .$$

The angle between two curves is the angle formed by the tangent lines at the point of intersection.

$$\theta = \tan^{-1} \frac{m_2 - m_1}{1 + m_1 m_2} .$$

Form a function that, given the slopes, computes the angle between two functions at a point of intersection.

[CLEAR] [MODE] [DEG]
« → a b 'ATAN((b-a)÷
(1+a×b [ENTER]

```
2:
1:  « → a b 'ATAN((b-a)/
     (1+a*b))' »
[STD] FIX  SCI  ENG [DEG] RAD
```

'ANG [STO]

```
3:
2:
1:
[STD] FIX  SCI  ENG [DEG] RAD
```

Lines have a constant slope. Read the slope for each directly from the given formula.

3 [ENTER]
−2 [ENTER]

```
3:
2:                          3
1:                         -2
[STD] FIX  SCI  ENG [DEG] RAD
```

Now compute the angle.

[USER] [ANG]

```
3:
2:
1:                         45
ANG
```

The lines intersect at an angle of 45°.

ANG is used in the next problem section.

Angle Between Two Curves

The angle of intersection for two curves is defined to be the angle formed by the tangent lines at the point of intersection. When an intersection point is located, the slopes of the functions at that point can be found. The problem is then that of two intersecting lines.

Example: Find the angle formed by the tangent lines at the points of intersection of the following functions.

$$F = 3x + 1$$

$$Y = 2x^2$$

Enter and save the given functions.

[CLEAR] [<>]
'3*X+1 [ENTER]

```
4:
3:
2:
1:                '3*X+1'
```

'F [STO]

```
4:
3:
2:
1:
```

'2*X^2 [ENTER]

```
4:
3:
2:
1:                '2*X^2'
```

'Y [STO]

```
4:
3:
2:
1:
```

Plot the two functions to obtain initial guesses at the points of intersection.

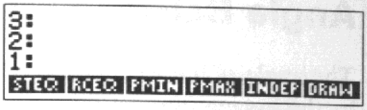
First, set the two functions equal to each other.

[USER] [Y] [F]
[=] [ENTER]

```
3:
2:
1:          '2*X^2=3*X+1'
  Y  |  F  | ANG |     |     |
```

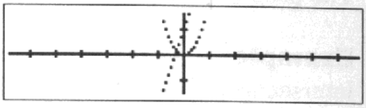
Store the equation.

PLOT STEQ



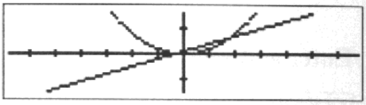
Clear the plot parameters and draw the equation with the two functions.

'PPAR PURGE
DRAW



Expand the height to see both intersection points.

ATTN 10 *H
DRAW



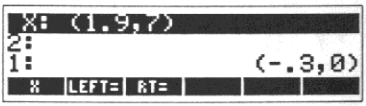
Digitize both intersection points. Enter the Solver to refine the guesses.

< ... < INS
> ... ^ INS
ATTN

SOLV SOLVR

Use the displayed value as an initial guess.

X

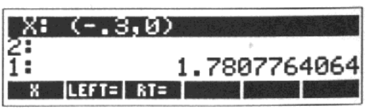


Compute a solution to the equation.

⬜ X

Repeat the procedure for the other point of intersection.

SWAP
X



[ ] X

```
X: -.280776406405
Sign Reversal
1:          -.280776406405
 X  LEFT= RT=
```

Recall *Y* to compute the slope at an intersection point.

[USER] Y

```
3:            1.7807764064
2:          -.280776406405
1:                '2*X^2'
 X  PPAR  EQ  Y  F  ANG
```

Take the derivative with respect to *x*.

'X [ENTER]
[d/dx]

```
3:            1.7807764064
2:          -.280776406405
1:          -1.12310562562
 X  PPAR  EQ  Y  F  ANG
```

Evaluate at one intersection point.

The last root computed remains assigned to *x*. The slope of the line can be read from the given expression.

3 [ENTER]

```
3:          -.280776406405
2:          -1.12310562562
1:                       3
 X  PPAR  EQ  Y  F  ANG
```

Use the ANG function to compute the angle.

ANG

```
3:            1.7807764064
2:          -.280776406405
1:          -60.1164404136
 X  PPAR  EQ  Y  F  ANG
```

This is in degrees.

Ready the stack to operate on the second intersection point.

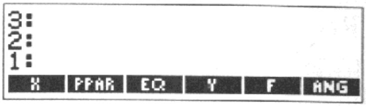
[DROP]
[DROP]

```
3:
2:
1:            1.7807764064
 X  PPAR  EQ  Y  F  ANG
```

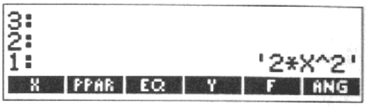
Compute the derivative of Y.

Assigning a numeric value to x at this point will mean a numeric value for the derivative when it is computed.

'X [STO]

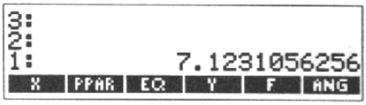


[Y]



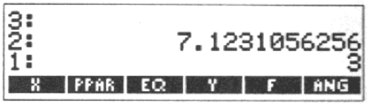
The derivative is computed with respect to x.

'X [ENTER]
[d/dx]



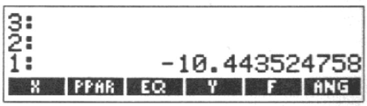
Enter the slope of the line.

3 [ENTER]



Again use the ANG function to compute the intersection angle.

[ANG]



Purge the variables created in the last two sections.

{F Y X ANG [ENTER] [PURGE]

Differential Calculus

This section includes problems of differential calculus, including function minimization, computing tangent lines, and several methods of implicitly differentiating functions. Several important features of the HP-28C are highlighted – creating user-defined derivatives, keyboard algebra for solving complex problems, and effective use of user flag 59 (the infinite result flag) and flag 35 (symbolic evaluation of constants).

Minimize Perimeter

Science, engineering, and business share the need to find the minimum values of given functions as some parameter changes. In this section, the function represents area and the parameter is the area's perimeter.

Example: To minimize material expense, find the mimimum amount of fencing required to enclose a rectangular plot measuring 200 square feet if one side is next to a building and needs no fence.

Let the sides be called x and y with y parallel to the building. The perimeter to be minimized is

$$P = 2x + y.$$

The area of the plot

$$x*y = 200$$

gives the relationship between x and y.

Clear the display and make certain variables X and Y have no assigned values. Clear flag 59 to ignore 'Infinite Result' errors while plotting.

[CLEAR] [MODE] 2 ≣FIX≣ [< >]
'X [PURGE]
'Y [PURGE]
59 CF [ENTER]

```
4:
3:
2:
1:
```

Enter the perimeter.

'2×X+Y [ENTER]

```
4:
3:
2:
1:                    '2*X+Y'
```

Enter the area.

'X×Y=200 [ENTER]

```
4:
3:
2:                    '2*X+Y'
1:                  'X*Y=200'
```

Isolate X.

'X [ALGEBRA] ≣ISOL≣

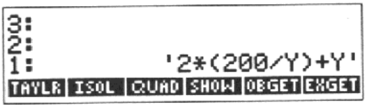
```
3:
2:                    '2*X+Y'
1:                    '200/Y'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Store the equation for X.

'X STO

Evaluate the expression for the perimeter.

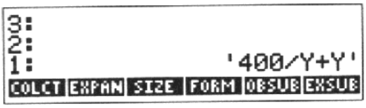
EVAL



This expresses the perimeter in terms of one variable.

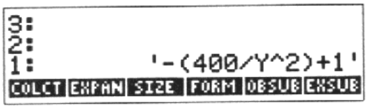
Collect terms.

COLCT



Compute the derivative. Roots of this will yield the mimimum value of Y.

'Y ENTER
d/dx

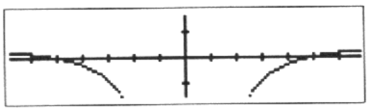


Plot the derivative to obtain a guess at the root.

PLOT STEQ
'PPAR PURGE
'Y INDEP

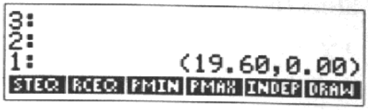
The steps below expand the plotting area and draw the graph. If you have no prior knowledge of the appearance of the graph, you may first wish to plot the graph, modify the plotting area accordingly, and then plot the graph a second time (i.e. DRAW ATTN, and *then* proceed with the steps below).

2 *H
4 *W
DRAW



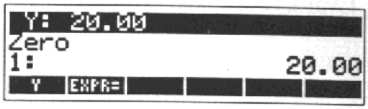
Digitize a seed for Y. Pick the guess near the positive root.

[>] . . . [>] [INS]
[ATTN]



Use the digitized value as a seed to compute Y.

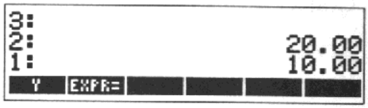
[SOLV] SOLVR
Y
[ ] Y



The side parallel to the building must be 20 feet long.

Recall and evaluate the expression for X.

X [ENTER]
[EVAL]



Forty feet of fencing is required (two ends ten feet long, and one side 20 feet long).

Purge the variables created in the example.

{X Y [ENTER] [PURGE]

Mimimize Surface Area

This section uses differential calculus to minimize surface area. An application of this solution is in manufacturing, where minimization can reduce wasted raw material and increase profit. Other problem specifications may, of course, add constraints or considerations to the final real-world solution.

Example: In the problem below, user flag 35 is set to maintain symbolic constants until the end of the solution.

Find the dimensions of a one liter can that has the minimum surface area.

The surface area of a can (a right circular cylinder) is

$$A = 2\pi R^2 + 2\pi RH \ .$$

The volume is

$$V = \pi R^2 H$$

where R is the radius and H is the height of the can. To minimize the surface area, the area is expressed in terms of either R or H and that expression is then differentiated with respect to that variable. Proceed by isolating H in the volume equation and finding the root of the derivative of the area taken with respect to R.

Clear the variables R, V, and H, and set flag 35.

Keys	Display
[CLEAR] [<>] {R V H [ENTER] [PURGE] 35 SF [ENTER]	4: 3: 2: 1:

Factor out $2\pi R$ and key in the expression for the surface area.

Keys	Display
'2×π×R×(R+H [ENTER]	4: 3: 2: 1: '2*π*R*(R+H)'

Duplicate the expression and store a copy for later use.

Keys	Display
[ENTER] 'A [STO]	4: 3: 2: 1: '2*π*R*(R+H)'

Enter the volume.

'V=π×R^2×H [ENTER]

```
4:
3:
2:              '2*π*R*(R+H)'
1:                'V=π*R^2*H'
```

Isolate *H*.

'H [ENTER]

```
4:
3:              '2*π*R*(R+H)'
2:                'V=π*R^2*H'
1:                        'H'
```

[ALGEBRA] [ISOL]

```
3:
2:              '2*π*R*(R+H)'
1:              'V/(π*R^2)'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Store it as *H*.

'H [STO]

```
3:
2:
1:              '2*π*R*(R+H)'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Now substitute for *H* in the area equation.

[EVAL]

```
2:
1:  '2*π*R*(R+V/(π*R^2))
    '
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Take the derivative with respect to *R*.

'R [ENTER]
[d/dx]

```
1:  '2*π*(R+V/(π*R^2))+2
    *π*R*(1-V*(π*(2*R))/
    (π*R^2)^2)'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Collect terms.

[COLCT]

```
1:  '2*(1-2*(R^2*π)^(-2)
    *R*V*π)*R*π+2*(R^(-2
    )*V/π+R)*π'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Prepare to plot the derivative to obtain a guess for the root.

[PLOT] [STEQ]

```
3:
2:
1:
STEQ RCEQ PMIN PMAX INDEP DRAW
```

One liter is the same as 1000 cubic centimeters. Enter the volume as 1000; the answer will be in centimeters.

1000 ENTER
'V STO

Purge the existing plot parameters and expand the plotting area.

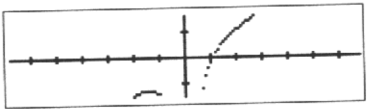
'PPAR PURGE
100 *H
5 *W

To find the radius that minimizes the area, specify *R* as the independent plotting variable. Clear flag 59 to ignore 'Infinite Result' errors that may occur while plotting.

R INDEP
59 CF ENTER

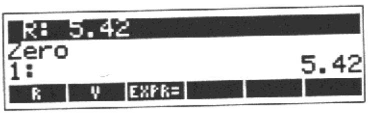
Draw the graph and digitize an initial guess for the Solver.

DRAW > . . . > INS



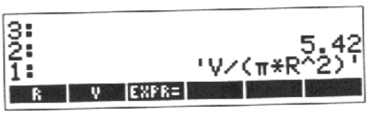
Now store the initial guess and compute the root.

ATTN SOLV SOLVR
R ⬜ R

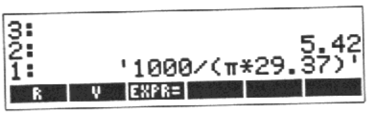


This is the radius. Now find the height.

'H ENTER
EVAL



EVAL



Compute the area.

A [EVAL]

```
2:       '1000/(π*29.37)'
1:  '2*π*5.42*(5.42+V/(π
    *R^2))'
 R  | V  | EXPR= |  |  |
```

Evaluate to a numerical result.

[EVAL]

```
2:       '1000/(π*29.37)'
1:  '2*π*5.42*(5.42+1000
    /(π*29.37))'
 R  | V  | EXPR= |  |  |
```

Reduce the expression to a real number.

[→NUM]

```
3:                  5.42
2:       '1000/(π*29.37)'
1:                553.58
 R  | V  | EXPR= |  |  |
```

To check that this is a minimum, compute the second derivative.

[SOLV] [RCEQ]
'R [ENTER]

```
3:                553.58
2:  '2*(1-2*(R^2*π)^(-2…
1:                   'R'
STEQ|RCEQ|SOLVR|ISOL|QUAD|SHOW
```

[d/dx]
[→NUM]

```
3:       '1000/(π*29.37)'
2:                553.58
1:                 37.70
STEQ|RCEQ|SOLVR|ISOL|QUAD|SHOW
```

The second derivative is positive; therefore the curve is concave up. The root is a local minimum.

Purge the variables created in this problem section.

{A H R V [ENTER] [PURGE]

Lines Tangent To A Circle

This section demonstrates manipulation of equations using the algebraic capabilities of the HP-28C. It is often necessary to compute the derivative of a function that cannot easily be expressed in terms of one variable. In this case we use implicit differentiation. This is the first of three methods for implicit differentiation shown in this booklet. Problem sections "Implicit Differentiation With User-Defined Derivative" and "Implicit Functions" show two other methods.

Example: Find the two points on a circle of radius 1 that have tangent lines passing through the point (2,2).

There are two expressions for the slope of the tangent lines – one from the circle itself and the other from the point exterior to the circle.

Clear the working variables to ensure a symbolic answer. This problem also demonstrates a simple error recovery procedure. To ensure that the recovery works, turn on UNDO.

```
[CLEAR]
{Y R B A EQ X [ENTER] [PURGE]
[MODE] [+UND]
```

```
3:
2:
1:
+CMD [-CMD] +LAST [-LAST] [+UND] -UND
```

The general equation for a circle is $x^2+y^2-r^2=0$, where r is the radius. Implicitly differentiate this equation.

Enter it for step by step differentiation. Note that "∂" is obtained by pressing the [d/dx] key after the ['] key.

```
'∂X(X^2+Y^2-R^2 [ENTER]
```

```
3:
2:
1:      '∂X(X^2+Y^2-R^2)'
+CMD [-CMD] +LAST [-LAST] [+UND] -UND
```

```
[EVAL]
```

```
2:
1:  '∂X(X^2+Y^2)-∂X(R^2)
    '
+CMD [-CMD] +LAST [-LAST] [+UND] -UND
```

Step through the derivative watching for the term representing the *dy* /*dx* term.

[EVAL]

```
2:
1:  '∂X(X^2)+∂X(Y^2)-∂X(
    R)*2*R^(2-1)'
+CMD -CMD +LAST -LAST +UND -UND
```

One more step-by-step differentiation will generate the *dy* /*dx* term from the ∂*X*(*Y*^2) term in the expression.

[EVAL]

```
2:
1:  '∂X(X)*2*X^(2-1)+∂X(
    Y)*2*Y^(2-1)'
+CMD -CMD +LAST -LAST +UND -UND
```

Now collect terms to shorten the expression.

[ALGEBRA] [COLCT]

```
2:
1:  '∂X(X)*2*X+∂X(Y)*2*Y
    '
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

This is a critical step. Replace the derivative sub-expression with a variable that can be isolated. Count all characters, except parentheses, up to and including the second partial derivative symbol. The derivative symbol is the ninth item for making the substitution.

9 [ENTER]
'DY [EXSUB]

```
3:
2:
1:     '∂X(X)*2*X+DY*2*Y'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Evaluate once more to clear the last derivative.

[EVAL]

```
3:
2:
1:             '2*X+DY*2*Y'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Solve for $\frac{dy}{dx}$.

'DY [ISOL]

```
3:
2:
1:             '-(2*X/Y/2)'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Collect the 2's.

COLCT

```
3:
2:
1:                '-(X/Y)'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

This is the slope of any line tangent to the circle. Tangent lines that pass through a point (A,B) exterior to the circle have slope $(y - B)/(x - A)$, where the point (x, y) is on the circle.

' (Y-B)÷(X-A ENTER

```
3:
2:                '-(X/Y)'
1:          '(Y-B)/(X-A)'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

This line must be a tangent to the circle; that is, the expressions for the slope must be equal.

= ENTER

```
3:
2:
1:  '-(X/Y)=(Y-B)/(X-A)'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Use algebra to solve for y.

Y ×

```
2:
1:  '-(X/Y*Y)=(Y-B)/(X-A
    )*Y'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Clear the denominators by collecting terms and multiplying through by denominator terms.

COLCT

```
2:
1:  '-X=INV(-A+X)*(-B+Y)
    *Y'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Extract the denominator term.

7 EXGET

```
3:
2:
1:                 '-A+X'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Since EXGET 'consumes' the original expression, a copy should have been made first. It is easy to recover from the error.

UNDO

```
2:
1:  '-X=INV(-A+X)*(-B+Y)
    *Y'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Make a copy and re-execute EXGET.

ENTER
7 EXGET

```
3:
2:  '-X=INV(-A+X)*(-B+Y…
1:                '-A+X'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Multiply through by the extracted term.

×

```
2:
1:  '-(X*(-A+X))=INV(-A+
    X)*(-B+Y)*Y*(-A+X)'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

The denominator is now cleared.

COLCT

```
2:
1:  '-((-A+X)*X)=(-B+Y)*
    Y'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

The following expansions distribute the x and y terms.

EXPAN

```
3:
2:
1:  '-(-A+X)*X=-B*Y+Y*Y'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

EXPAN

```
3:
2:
1:     '(A-X)*X=-B*Y+Y*Y'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

EXPAN

```
3:
2:
1:     'A*X-X*X=-B*Y+Y*Y'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Now collect terms.

COLCT

```
3:
2:
1:     '-X^2+A*X=Y^2-B*Y'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Gather like powers.

First gather powers of 2.

[ENTER]
1 [EXGET]

```
3:
2:    '-X^2+A*X=Y^2-B*Y'
1:                '-X^2'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

[-]

```
2:
1:  '-X^2+A*X+X^2=Y^2-B*
    Y+X^2'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

[COLCT]

```
3:
2:
1:  'A*X=-(B*Y)+X^2+Y^2'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Now gather powers of 1.

[ENTER]
7 [EXGET]

```
3:
2:  'A*X=-(B*Y)+X^2+Y^2'
1:                'B*Y'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

[+]

```
2:
1:  'A*X+B*Y=-(B*Y)+X^2+
    Y^2+B*Y'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

[COLCT]

```
3:
2:
1:     'A*X+B*Y=X^2+Y^2'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

The right hand side of this equation is r^2. Make a substitution for the right hand side.

12 [ENTER]
'R^2 [EXSUB]

```
3:
2:
1:          'A*X+B*Y=R^2'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

This linear equation can now be solved for y.

Y [ISOL]

```
3:
2:
1:          '(R^2-A*X)/B'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Save this for later use.

'Y [STO]

```
3:
2:
1:
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Enter the equation for the circle.

'X^2+Y^2-R^2 [ENTER]

```
3:
2:
1:          'X^2+Y^2-R^2'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Substitute in the expression for y.

[EVAL]

```
2:
1: 'X^2+((R^2-A*X)/B)^2
   -R^2'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

This is a quadratic equation for x, and is easy to solve.

'X [QUAD]

```
1: '(A/B*2*(R^2/B)+s1*√
   ((-(A/B*2*(R^2/B)))^
   2-4*((2-A/B*2*(-(A/B
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Shorten it by collecting terms.

[COLCT]

```
1: '(√(-(2*(2+2*A^2*B^(
   -2))*((INV(B)*R^2)^2
   -R^2))+(-(2*A*B^(-2)
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Duplicate and store this expression for x.

[ENTER] 'X [STO]

```
1: '(√(-(2*(2+2*A^2*B^(
   -2))*((INV(B)*R^2)^2
   -R^2))+(-(2*A*B^(-2)
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

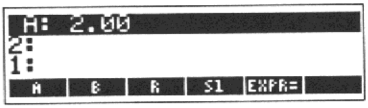
In the Solver, you can assign the numbers needed to complete the given problem.

[SOLV] [STEQ]
[SOLVR]

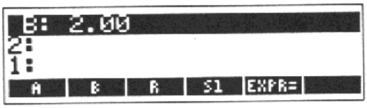
```
3:
2:
1:
A B R S1 EXPR=
```

The exterior point is (2,2).

2 A

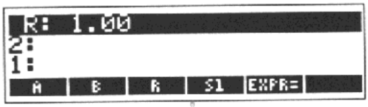


2 B



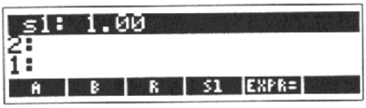
The radius of the circle is 1.

1 R



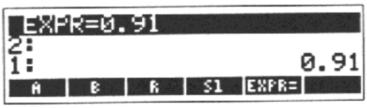
There are two roots, one for each point on the circle.

1 S1



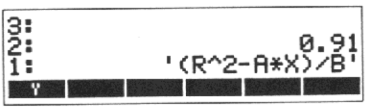
Solve for the x coordinate.

EXPR=

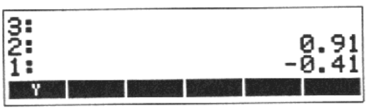


Now solve for the y coordinate.

USER Y

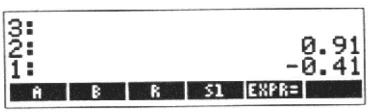


→NUM

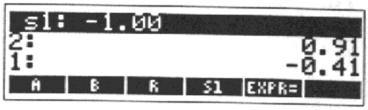


Repeat the process for the other point.

SOLV SOLVR

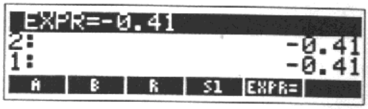


-1 ≣S1≣



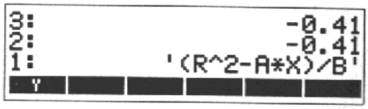
Solve for the *x* coordinate.

≣EXPR=≣

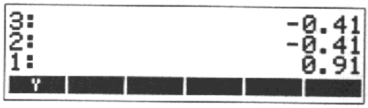


Now compute the *y* coordinate.

[USER] ≣Y≣



[→NUM]



The points of tangency are (0.91, −0.41) and (−0.41,0.91).

The general solution approach solves the problem for any circle and any exterior point.

Purge the variables created in this problem section.

{X Y A B R s1 [ENTER] [PURGE]

Implicit Differentiation With User-Defined Derivative

This section uses a user-defined derivative for implicit differentiation of a function. Refer to the Reference Manual for additional information.

Example: Given the equation $\sqrt{x} + \sqrt{y} = 3$, express $\frac{dy}{dx}$ in terms of x and y.

Create a user-defined derivative for the function $y(x)$. User-defined derivatives must take two inputs from the stack; the definition below simply discards them and returns the variable *DY*, which can be isolated.

[CLEAR] [<>]

« → x dx 'DY [ENTER]

```
4:
3:
2:
1:          « → x dx 'DY' »
```

Store it in the variable *derY*.

'derY [STO]

```
4:
3:
2:
1:
```

Enter the *Y* variable as a function of *X*.

'√X + √Y(X) - 3 [ENTER]

```
4:
3:
2:
1:              '√X+√Y(X)-3'
```

Differentiate with respect to *X*.

'X [ENTER] [d/dx]

```
3:
2:
1:  'INV(2*√X)+DY/(2*√Y(
    X))'
```

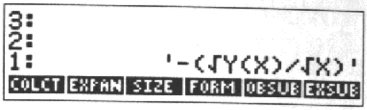
Solve for *DY*. Remember that *DY* represents $\frac{dy}{dx}$.

'DY [ALGEBRA] [ISOL]

```
2:
1:  '-(INV(2*√X)*(2*√Y(X
    )))'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Simplify to get the solution.

COLCT



Purge the user-defined derivative created in this example.

`'derY` PURGE

Taylor Series Error Term

Many physics and engineering problems are made solvable by expanding non-linear terms in a Taylor series. Ignoring the quadratic and higher degree terms leads to an approximate solution that is good for 'small displacement'. This problem shows how to find the range for which the error in a Taylor series expansions stays small.

Example: Find the range of x for which the error in the 3rd degree approximation of $\sin(x)$ is less than .1.

The Taylor Series error term is

$$R_n(x) = f^{(n+1)}(c)\frac{x^{n+1}}{(n+1)!}$$

The exponent of f indicates the order of differentiation.

It is important to recognize that the error is the next term in the expansion. Since the 'sin' function contains only odd powered terms, look at the difference in the 5th and 3rd degree approximations. For the 'sin' function the $n+1$ derivative has a maximum of 1.

Thus $R_{(n+1)} < \frac{x^{n+1}}{(n+1)!}$.

Compute the 5th degree expansion.

Set the angle mode. Key in the function and the variable name.

[CLEAR] [MODE] RAD
'SIN(X [ENTER]
X [ENTER]

```
3:
2:                'SIN(X)'
1:                    'X'
 STD [ FIX ] SCI  ENG  DEG [ RAD ]
```

Key in the order and find the Taylor Series.

5 [ALGEBRA] TAYLR

```
2:
1: 'X-0.17*X^3+0.01*X^5
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Now compute the 3rd degree approximation.

'SIN(X [ENTER]
X [ENTER]
3 TAYLR

```
3:
2: 'X-0.17*X^3+0.01*X^…
1:          'X-0.17*X^3'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Make a copy and store this result for later use.

[ENTER] 'APS [STO]

```
3:
2: 'X-0.17*X^3+0.01*X^…
1:           'X-0.17*X^3'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Subtract the two approximations.

[-]

```
2:
1: 'X-0.17*X^3+0.01*X^5
   -(X-0.17*X^3)'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Collect terms. The remaining expression is the 3rd degree error term.

COLCT

```
3:
2:
1:               '0.01*X^5'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Set it equal to .1 and then solve for x.

.1 [ENTER] [=] [ENTER]

```
3:
2:
1:          '0.01*X^5=0.10'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

There are several ways to solve for x. The ISOL command will isolate x in the displayed equation, and result in a generalized expression for x. A second approach is to use Solver to compute x. A third approach would be to use the laws of algebra and the capabilities of the HP-28C and solve for x 'long-hand'. All three methods are shown below; the third approach is included to illustrate the power of FORM in the ALBEGRA menu.

Choose any one of the three methods which follow, and then proceed to the "Conclusion" portion of this problem.

Method 1: Using ISOL

Find the generalized expression for x. The status of flags 34 and 35 will affect the next display. The expression below is the result with both flags 34 and 35 clear. Refer to the Reference Manual for a discussion on alternate settings of these flags. With flag 34 set, you would immediately obtain the result 1.64 found after the next several steps.

'X [ISOL]

```
2:
1:  'EXP((0.00,6.28)*n1/
    5)*1.64'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Assign a value of zero to the arbitrary integer *n1* introduced into the isolation of the variable x.

0 [ENTER]
'n1 [STO]

```
2:
1:  'EXP((0.00,6.28)*n1/
    5)*1.64'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Evaluate the expression.

[EVAL]

```
3:
2:
1:           (1.64,0.00)
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Extract the real component of the complex result.

[REAL] [ABS]

```
3:
2:
1:                  1.64
ABS SIGN MANT XPON
```

Now skip to the discussion and keystrokes labeled "Conclusion" to complete this problem.

Method 2: Using Solver

This method illustrates a simple approach to solve for x with the Solver.

Proceed to the Solver menu and store the equation.

[SOLV] STEQ
SOLVR

```
3:
2:
1:
 X | LEFT= | RT= |  |  | 
```

Solve for the variable x.

[ ] X

```
X: 1.64
Sign Reversal
1:                  1.64
 X | LEFT= | RT= |  |  | 
```

Now skip to the discussion and keystrokes labeled "Conclusion" to complete this problem.

Method 3: Using FORM and algebraic manipulation

This method illustrates the use of FORM and the keyboard capabilities of the HP-28C to manipulate algebraic expressions. While the two methods above are more direct, this alternative follows a traditional 'paper-and-pencil' approach towards the solution.

First, compute the fifth root of the equation.

'1÷5 [ENTER] [^]

```
2:
1:  '(0.01*X^5)^(1/5)=
     0.10^(1/5)'
COLCT|EXPAN|SIZE|FORM|OBSUB|EXSUB
```

In FORM, first distribute the left hand exponential, and then associate the 5 and 1/5. Then collect terms in the expression.

FORM

```
(((0.01*(X^5))^(1/5))=(
0.01^(1/5)))
COLCT|EXPAN|LEVEL|EXGET| [←] | [→]
```

Move to the exponentiation sign.

[→] . . . [→]

```
(((0.01*(X^5))^(1/5))=(
0.01^(1/5)))
COLCT|EXPAN|LEVEL|EXGET| [←] | [→]
```

Distribute the left-hand exponential.

[←D]

```
(((0.01^(1/5))*((X^5)^(
1/5)))=(0.01^(1/5)))
 1/()  ←→   ←D   D→   ←A   A→
```

Move to the second exponentiation sign.

[[→]] . . . [[→]]

```
(((0.01^(1/5))*((X^5)^(
1/5)))=(0.01^(1/5)))
COLCT EXPAN LEVEL EXGET [←] [→]
```

Now associate the 5 and 1/5 in the expression.

[A→]

```
(((0.01^(1/5))*(X^(5*(1
/5))))=(0.01^(1/5)))
 1/()  E()  ←D   D→   ←A   A→
```

Exit FORM and collect terms.

[ATTN] [COLCT]

```
3:
2:
1:            '0.38*X=0.63'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Solve for *x*.

'X [ISOL]

```
3:
2:
1:                     1.64
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Conclusion: The variable *x* has now been isolated by one of the three methods described above. Proceed with the remainder of this problem solution.

The 'sin' is symmetric so $R^3 < .1$ for $-1.64 < x < 1.64$. Check the result in Solver.

[USER] [APS]

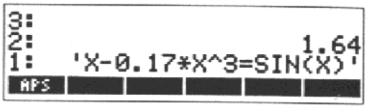
```
3:
2:                     1.64
1:             'X-0.17*X^3'
 APS
```

Compare the approximation to sin(x).

'SIN(X [ENTER]

```
3:                     1.64
2:             'X-0.17*X^3'
1:                 'SIN(X)'
 APS
```

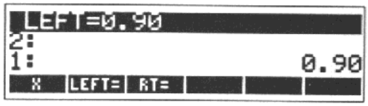
[=] [ENTER]



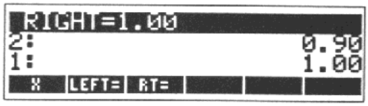
[SOLV] [STEQ] [SOLVR]

[X]

[LEFT=]

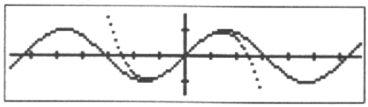


[RT=]



Clearly the difference is .1. Now plot the two equations. Purge the current plot parameters and draw the function.

[PLOT] 'PPAR [PURGE] [DRAW]



If the Taylor series approximation is needed for values of x that differ significantly from 0, the center of the expansion should be shifted, as demonstrated in the tangent line problem in the next section.

Purge the variables created in this problem section.

{X APS EQ [ENTER] [PURGE]

Tangent Lines and Taylor Series

This section demonstrates how to use the first order Taylor series to generate a tangent line equation. The example problem expands about a point other than the origin.

Example: Find the equation of the line tangent to the sine curve at $X = 1$.

Clear the stack. The first degree polynomial Taylor series expansion is the tangent line at the point of expansion.

Enter the function to be expanded.

Keystrokes	Display
[CLEAR] [<>] 'SIN(X [ENTER]	4: 3: 2: 1: 'SIN(X)'

Change the variable to correspond with the new center. That is, $Y = 0$ corresponds to $X = 1$.

Keystrokes	Display
'Y+1 [ENTER]	4: 3: 2: 'SIN(X)' 1: 'Y+1'
'X [STO]	4: 3: 2: 1: 'SIN(X)'

This is the function to be expanded.

Keystrokes	Display
[EVAL]	4: 3: 2: 1: 'SIN(Y+1)'

Enter the variable and the degree of the polynomial.

Keystrokes	Display
'Y [ENTER] 1 [ENTER]	4: 3: 'SIN(Y+1)' 2: 'Y' 1: 1.00

Find the Taylor expansion.

[ALGEBRA] TAYLR

```
3:
2:
1:          '0.84+0.54*Y'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

This is the equation in Y.

[USER] X

```
3:
2:          '0.84+0.54*Y'
1:                  'Y+1'
  X
```

Recall the change of variable equation.

'X [ENTER]
[=] [ENTER]

```
3:
2:          '0.84+0.54*Y'
1:                'Y+1=X'
  X
```

Clear the original variable change equation.

'X [PURGE]

```
3:
2:          '0.84+0.54*Y'
1:                'Y+1=X'
ORDER CLUSR MEM
```

Solve for Y.

'Y [ENTER]
[ALGEBRA] ISOL

```
3:
2:          '0.84+0.54*Y'
1:                  'X-1'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Save the expression for Y.

'Y [STO]

```
3:
2:
1:          '0.84+0.54*Y'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Change back to the original variable and simplify the resulting expression.

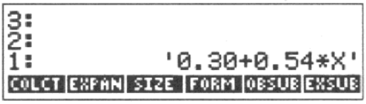
[EVAL]

```
3:
2:
1:      '0.84+0.54*(X-1)'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

EXPAN

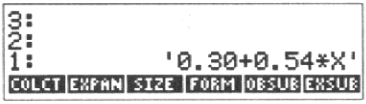
```
2:
1:  '0.84+(0.54*X-0.54*1
    )'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

COLCT



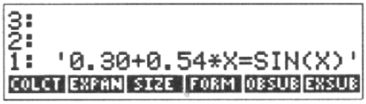
Save a copy of this expression for the next problem section.

[ENTER]
'STN [STO]

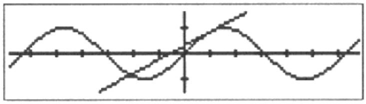


Plot the two equations for a quick check.

'SIN(X [ENTER]
[=] [ENTER]



[PLOT] STEQ
'PPAR [PURGE]
'X INDEP
DRAW



Purge variables X and Y for the next problem section.

[ATTN] 'X [PURGE] 'Y [PURGE]

Normal Line

In the previous section, the equation for the line came as a result of a Taylor series expansion. This section continues by manually assembling the expression for the normal line.

Example: Compute the equation of the line normal (perpendicular) to the sine curve at $x = 1$.

First recall the equation for the tangent line.

[CLEAR]
[USER] STN
[ENTER] [ENTER]

```
3:         '0.30+0.54*X'
2:         '0.30+0.54*X'
1:         '0.30+0.54*X'
PPAR  EQ  STN
```

We need the value of the function at $x = 1$. Evaluate the expression.

1 [ENTER]
'X [STO]
[EVAL]

```
3:         '0.30+0.54*X'
2:         '0.30+0.54*X'
1:                  0.84
X  PPAR  EQ  STN
```

This is Y_0.

Since we want symbolic solutions purge the value of x.

'X [PURGE]

```
3:         '0.30+0.54*X'
2:         '0.30+0.54*X'
1:                  0.84
PPAR  EQ  STN
```

The general point slope formula for a line is

$$Y - Y_0 = m\,(X - X_0)\ .$$

Y_0 is on the stack. Form the left hand side of the relationship above.

Y [ENTER]
[SWAP]
[−]

```
3:         '0.30+0.54*X'
2:         '0.30+0.54*X'
1:             'Y-0.84'
PPAR  EQ  STN
```

Now form the right hand side. Bring the original line in position to find the slope.

[SWAP]
'X [ENTER]

```
3:             'Y-0.84'
2:         '0.30+0.54*X'
1:                  'X'
PPAR  EQ  STN
```

Find the slope by taking the derivative.

[d/dx]

```
3:          '0.30+0.54*X'
2:               'Y-0.84'
1:                  0.54
PPAR  EQ  STN
```

This is the slope of the tangent line. The slope of the normal line is

$$m_n = -\frac{1}{m_t} .$$

Compute m_n.

[CHS]
[1/x]

```
3:          '0.30+0.54*X'
2:               'Y-0.84'
1:                 -1.85
PPAR  EQ  STN
```

Now compute the right hand side.

'X-1 [ENTER]

```
3:               'Y-0.84'
2:                 -1.85
1:                 'X-1'
PPAR  EQ  STN
```

[×]

```
3:          '0.30+0.54*X'
2:               'Y-0.84'
1:        '-(1.85*(X-1))'
PPAR  EQ  STN
```

Form the entire equation.

[=] [ENTER]

```
2:          '0.30+0.54*X'
1:  'Y-0.84=-(1.85*(X-1)
    )'
PPAR  EQ  STN
```

Solve for Y.

'Y [ENTER]
[ALGEBRA] [ISOL]

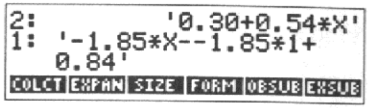
```
3:
2:          '0.30+0.54*X'
1:  '-(1.85*(X-1))+0.84'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Simplify the expression.

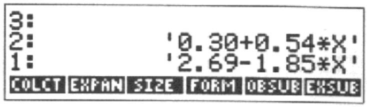
[EXPAN]

```
3:
2:          '0.30+0.54*X'
1:    '-1.85*(X-1)+0.84'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

EXPAN

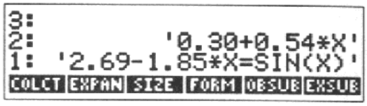


COLCT

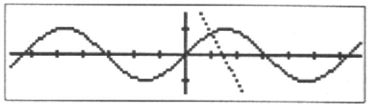


Plot the resulting function.

'SIN(X [ENTER]
[=] [ENTER]



[PLOT] STEQ
'PPAR [PURGE]
'X INDEP
DRAW



Purge the following variables created in this section.

[ATTN] {STN EQ PPAR [ENTER] [PURGE]

Implicit Functions

The Implicit Function Theorem is, perhaps, the most elegant of three methods shown for implicit differentiation. This section demonstrates a more general method for finding the equation of a line than the previous sections.

Example: Find the equation of the line tangent to the function $x^2+xy-3=0$ at $x=1$.

Begin by defining a function to compute the derivative of a general function $F(x,y)$. The formula, a result of the implicit function theorem, can be used as long as $\frac{\partial F}{\partial y}\neq 0$ holds.

Purge the variables to be used to ensure symbolic solutions.

`{X Y y x` [ENTER] [PURGE] [<>]

```
4:
3:
2:
1:
```

Enter the function for computing implicit derivatives.

`« → a '-∂X(a)÷∂Y(a`
[ENTER]

```
3:
2:
1: « → a '-∂X(a)/∂Y(a)'
   »
```

Store the implicit derivatives function.

`'IMP` [STO]

```
4:
3:
2:
1:
```

Enter and store the general formula for a line.

`'y=m×(x-X)+Y` [ENTER]

```
4:
3:
2:
1:            'y=m*(x-X)+Y'
```

`'LINE` [STO]

```
4:
3:
2:
1:
```

The function must be expressed in terms of X and Y due to the use of those variables in the function IMP.

'X^2+X×Y−3 [ENTER]

```
4:
3:
2:
1:            'X^2+X*Y-3'
```

'F [STO]

```
4:
3:
2:
1:
```

Now find $\frac{dy}{dx}$ in terms of X and Y.

[USER] [F]

```
3:
2:
1:            'X^2+X*Y-3'
  F  LINE  IMP
```

[IMP]

```
2:
1:  '-(∂X(X^2+X*Y)/∂Y(X^
    2+X*Y))'
  F  LINE  IMP
```

Evaluate the expression until all the partial derivative symbols are gone.

[EVAL]

```
2:
1:  '-((∂X(X^2)+∂X(X*Y))
    /(∂Y(X^2)+∂Y(X*Y)))'
  F  LINE  IMP
```

[EVAL]

```
1:  '-((∂X(X)*2*X^(2-1)+
    (∂X(X)*Y+X*∂X(Y)))/(
    ∂Y(X)*2*X^(2-1)+(∂Y(
  F  LINE  IMP
```

[EVAL]

```
3:
2:
1:          '-((2*X+Y)/X)'
  F  LINE  IMP
```

This expression for the slope of $F(x, y)$ at any point on the curve must be the slope of the tangent line.

'm [STO]

```
3:
2:
1:
  M  F  LINE  IMP
```

Now determine the value of Y that corresponds to $x = 2$.

F

```
3:
2:
1:          'X^2+X*Y-3'
 M | F | LINE | IMP |  |
```

SOLV STEQ
SOLVR

```
3:
2:
1:
 X | Y | EXPR= |  |  |
```

2 X

```
X: 2.00
2:
1:
 X | Y | EXPR= |  |  |
```

EXPR=

```
EXPR='4+2*Y-3'
2:
1:             '4+2*Y-3'
 X | Y | EXPR= |  |  |
```

Solve for Y.

'Y ALGEBRA ISOL

```
3:
2:
1:                 -0.50
TAYLR | ISOL | QUAD | SHOW | OBGET | EXGET
```

'Y STO

```
3:
2:
1:
TAYLR | ISOL | QUAD | SHOW | OBGET | EXGET
```

With the coordinates of the point at the tangent line and the slope of the line in terms of those coordinates, evaluate and simplify the formula for the line.

USER LINE

```
3:
2:
1:          'y=m*(x-X)+Y'
 Y | X | EQ | M | F | LINE
```

EVAL

```
2:
1:  'y=-((2*X+Y)/X*(x-2)
    )-0.50'
 Y | X | EQ | M | F | LINE
```

[EVAL]

```
2:
1: 'y=-(1.75*(x-2))-
   0.50'
 Y | X | EQ | M | F | LINE
```

Use EXPAN to distribute the constant.

[ALGEBRA] [EXPAN]

```
3:
2:
1: 'y=-1.75*(x-2)-0.50'
COLCT|EXPAN|SIZE|FORM|OBSUB|EXSUB
```

[EXPAN]

```
2:
1: 'y=-1.75*x--1.75*2-
   0.50'
COLCT|EXPAN|SIZE|FORM|OBSUB|EXSUB
```

Finally, simplify the equation for the tangent line.

[COLCT]

```
3:
2:
1:          'y=3-1.75*x'
COLCT|EXPAN|SIZE|FORM|OBSUB|EXSUB
```

Purge the variables created in this problem section.

`{Y X EQ M F LINE IMP` [ENTER] [PURGE]

Integral Calculus

This section solves a number of problems of integral calculus, including integration of simple differential equations and computation of arc lengths, surfaces, and volumes. Both symbolic and numerical solutions are demonstrated with appropriate use of system flags.

Integration and Free Falling Body

This problem section demonstrates derivation of standard equations of motion through simple integration. The importance of the constant of integration is made clear, and how that constant is incorporated into the solution provided by the HP-28C.

Example: A stone is dropped from a bridge 100 ft above the water. Compute how long it takes to reach the water and its final velocity.

From Newton's 2nd law

$$F = m\ddot{x} \ .$$

The only force acting on a falling body is that of gravity.

$$F = -mg$$

Combining these,

$$\ddot{x} = -g \ .$$

This is the equation of motion for a freely falling body. A well-posed problem requires two initial conditions, the starting position and velocity. The problem then may be solved by integration.

This solution approach plots the final equation to facilitate root finding. Start by configuring the plot parameters.

CLEAR PLOT
'PPAR PURGE
100 *H
(0,-70 PMIN

Plot the displacement as a function of time. Let TM represent the time.

'TM INDEP

Start by integrating the above equation. Let GRV be the acceleration due to gravity. Since the expression to be integrated includes no 'TM' terms, the specified degree of the polynomial is zero.

'-GRV [ENTER]
'TM [ENTER]
0 [ENTER]

```
3:                    '-GRV'
2:                      'TM'
1:                      0.00
STEQ RCEQ PMIN PMAX INDEP DRAW
```

[∫]

```
3:
2:
1:              '-(GRV*TM)'
STEQ RCEQ PMIN PMAX INDEP DRAW
```

This is an expression for the velocity. At TM = 0 the initial velocity is V0.

V0 [+]

```
3:
2:
1:           '-(GRV*TM)+V0'
STEQ RCEQ PMIN PMAX INDEP DRAW
```

Store this for future use.

'VEL [STO]

```
3:
2:
1:
STEQ RCEQ PMIN PMAX INDEP DRAW
```

Now recall the velocity and prepare for a second integration. The integrand includes 'TM' to the first degree, so a '1' is specified for the last parameter to the integration.

[USER] [VEL]
'TM [ENTER]
1 [ENTER]

```
3:           '-(GRV*TM)+V0'
2:                      'TM'
1:                      1.00
VEL  PPAR
```

[∫]

```
3:
2:
1:     'V0*TM-GRV/2*TM^2'
VEL  PPAR
```

This is an expression for the displacement. At TM = 0, $x = X0$.

X0 [+]

```
2:
1:  'V0*TM-GRV/2*TM^2+X0
    '
VEL  PPAR
```

To put this in the standard form, use the expression manipulation capabilities in FORM.

[ALGEBRA] [FORM]

```
((( V0*TM)-((GRV/2)*(TM^
2)))+X0)
COLCT EXPAN LEVEL EXGET [←] [→]
```

Move the cursor to the minus sign.

[→] . . . [→]

```
(((V0*TM)-((GRV/2)*(TM^
2)))+X0)
COLCT EXPAN LEVEL EXGET [←] [→]
```

Commute the expressions about the minus sign.

[←→]

```
((-((GRV/2)*(TM^2))+(V0
*TM))+X0)
-() ↔ ←M M→ ←A A→
```

Exit FORM, make a copy, and save the expression for distance.

[ATTN] [ENTER]
'DST [STO]

```
2:
1:  '-(GRV/2*TM^2)+V0*TM
    +X0'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Store the expression for use in the Solver menu.

[SOLV] [STEQ]
[SOLVR]

```
3:
2:
1:
GRV TM V0 X0 EXPR=
```

In English units the acceleration due to gravity is 32 ft/sec/sec.

32 [GRV]

```
GRV: 32.00
2:
1:
GRV TM V0 X0 EXPR=
```

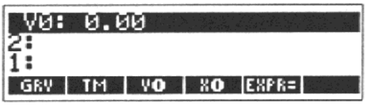
The bridge is 100 feet high.

100 [X0]

```
X0: 100.00
2:
1:
GRV TM V0 X0 EXPR=
```

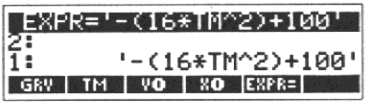
Since the stone is dropped, the initial velocity is zero.

0 [VO]



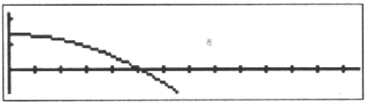
Evaluate the expression EQ.

[EXPR=]



To find the time required to hit the water, find a root of this equation. Digitize an initial guess from a plot of the equation.

[PLOT] [DRAW]
[↓] . . . [<] [INS]

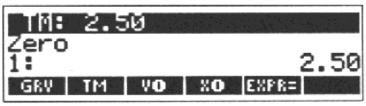


Assign the seed to TM.

[ATTN] [SOLV] [SOLVR]
[TM]

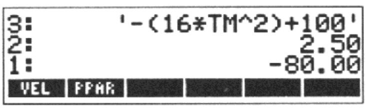
Solve for TM.

[ ] [TM]



The stone hits the water after 2.5 seconds. To find the velocity, recall VEL and evaluate it.

[USER] [VEL]
[EVAL]



The stone is falling at 80 feet per second.

By changing the initial conditions, the equations of motion developed in the previous example can be applied to a rock thrown straight up.

Example: A stone is thrown straight up from ground level with an initial velocity of 70 feet per second.

Compute its peak, the time elapsed until it hits the ground, and its final velocity.

Fetch the general equation for distance traveled.

[CLEAR]
[USER] [DST]

```
2:
1: '-(GRV/2*TM^2)+V0*TM
   +X0'
 TM | V0 | X0 | GRV | EQ | DST
```

Enter the SOLV menu and store the equation for analysis.

[SOLV] [STEQ]
[SOLVR]

```
3:
2:
1:
 GRV | TM | V0 | X0 | EXPR= |
```

The initial position is ground level or $x = 0$.

0 [X0]

```
X0: 0.00
2:
1:
 GRV | TM | V0 | X0 | EXPR= |
```

The initial velocity is 70 feet per second upward, and therefore positive.

70 [V0]

```
V0: 70.00
2:
1:
 GRV | TM | V0 | X0 | EXPR= |
```

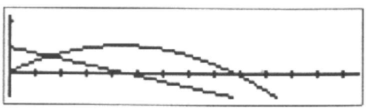
The plot parameters were set in the previous problem. Plot both the velocity and the distance equations.

[USER] [DST]
[VEL]
[=] [ENTER]

```
2:
1: '-(GRV/2*TM^2)+V0*TM
   +X0=-(GRV*TM)+V0'
 VEL | PPAR |  |  |  |
```

Store the equation for plotting.

[PLOT] [STEQ]
[DRAW]



The velocity is the first derivative of the distance; therefore the root of the velocity equation corresponds to a maximum of the distance equation. Digitize the roots of the velocity (where the straight line crosses the x-axis) and the distance (where the curve crosses the x-axis for the second time).

[<] . . . [↓] [INS]
[>] . . . [>] [INS]
[ATTN]

```
3:
2:           (2.30,-3.23)
1:           (4.45,-3.23)
STEQ RCEQ PMIN PMAX INDEP DRAW
```

Recall the equation for velocity and save the equation for analysis.

[USER] VEL

```
3:           (2.30,-3.23)
2:           (4.45,-3.23)
1:        '-(GRV*TM)+V0'
VEL PPAR
```

[SOLV] STEQ
SOLVR

```
3:
2:           (2.30,-3.23)
1:           (4.45,-3.23)
GRV TM V0 EXPR=
```

Enter the initial guess for the root and solve for TM.

[SWAP] TM [ ] TM

```
TM: 2.19
Zero
1:                   2.19
GRV TM V0 EXPR=
```

After 2.19 seconds, the stone reaches a maximum height. Recall the distance equation from the User menu and evaluate to find this height.

[USER] DST

```
2:                   2.19
1: '-(GRV/2*TM^2)+V0*TM
   +X0'
TM V0 X0 GRV EQ DST
```

[EVAL]

```
3:           (4.45,-3.23)
2:                   2.19
1:                  76.56
TM V0 X0 GRV EQ DST
```

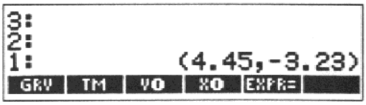
The rock reaches a height of 76.56 feet.

Now drop two numbers from the stack and fetch the distance equation for analysis.

[DROP]
[DROP]
DST

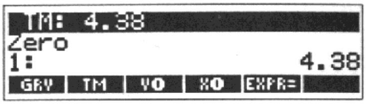
```
2:           (4.45,-3.23)
1: '-(GRV/2*TM^2)+V0*TM
   +X0'
TM V0 X0 GRV EQ DST
```

SOLV ≡ STEQ ≡
≡ SOLVR ≡



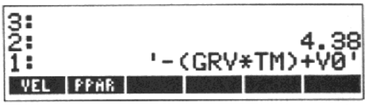
Enter the guess and solve for the root.

≡ TM ≡ □ ≡ TM ≡

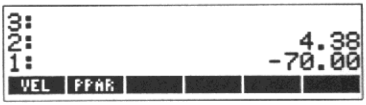


The rock hits after 4.38 seconds. Note that this is exactly twice the time required to reach the maximum height. Therefore the time spent going up is equal to the time spent falling back to the ground. To find the final velocity recall the velocity equation and evaluate.

USER ≡ VEL ≡



EVAL



Note that this number differs from the initial velocity in sign only. The rock's final speed is the same as its initial speed, but it is traveling in the opposite direction.

Purge the variables created in this problem section.

`{TM EQ VEL DST GRV X0 V0 PPAR` ENTER PURGE

Double Integration

This section uses both symbolic and numerical integration to solve common problems of integral calculus.

Example: Compute the area between the line

$$Y = x$$

and the parabola

$$Y = x^2 .$$

The area may be found by computing the double integral $\int_1^0 \int_{x^2}^{x} dy\ dx$.

To insure a symbolic answer purge the constant and the variable of integration.

[CLEAR]
{C Y [ENTER] [PURGE] [<>]

```
4:
3:
2:
1:
```

The next four displays show the calculator steps to compute $\int c\ dy$ where

$c = 1$. Because the result is simply y, you can choose to skip directly to the evaluation of the integral at its limits if you wish. If so, simply enter Y, and proceed to the steps below beginning with "Enter the upper limit".

Otherwise, prepare the stack for a symbolic integration with a first degree result. Start by integrating a constant.

'C [ENTER]
'Y [ENTER]
1 [ENTER]

```
4:
3:                 'C'
2:                 'Y'
1:                1.00
```

Execute the integral.

[∫]

```
4:
3:
2:
1:               'C*Y'
```

Eliminate the constant by equating it to 1.

1 [ENTER]
'C [STO]

```
4:
3:
2:
1:                'C*Y'
```

[EVAL]

```
4:
3:
2:
1:                  'Y'
```

Enter the upper limit.

'X [ENTER]
'Y [STO]

```
4:
3:
2:
1:                  'Y'
```

Save a copy of the integrand for later use and evaluate the integral at the limit.

[ENTER]
[EVAL]

```
4:
3:
2:                  'Y'
1:                  'X'
```

Repeat the process for the lower limit.

'X^2 [ENTER]
'Y [STO]

```
4:
3:
2:                  'Y'
1:                  'X'
```

Place a copy of the integrand in position for evaluation at the lower limit.

[SWAP]
[EVAL]

```
4:
3:
2:                  'X'
1:                'X^2'
```

The difference is the integrand for the second integration.

[-]

```
4:
3:
2:
1:              'X-X^2'
```

Key in the parameters for the integration.

{X 0 1 [ENTER]

```
4:
3:
2:                'X-X^2'
1:        { X 0.00 1.00 }
```

Key in the error bound.

.005 [ENTER]

```
4:
3:                'X-X^2'
2:        { X 0.00 1.00 }
1:                   0.01
```

Evaluate the second integral. The error bound provides accuracy to the number of displayed digits (assuming 2 [FIX]).

[∫]

```
4:
3:
2:                   0.17
1:               8.37E-4
```

The area is 0.17.

Purge the variables created in this problem section.

{Y C [ENTER] [PURGE]

Area Between Two Curves

This section provides a general approach for finding the area between any two intersecting curves.

Example: Find the area inclosed by the parabola $f(x)=x^2$ and the line $y(x)=x+3$.

The area between two curves can be found by computing the integral $\int_a^b |f(x)-y(x)|\,dx$. In this problem the limits will be the intersection points of the curves.

Enter and store the integrand.

[CLEAR] [<>]
'ABS(F-Y [ENTER]

```
4:
3:
2:
1:              'ABS(F-Y)'
```

'AREA [STO]

```
4:
3:
2:
1:
```

Enter and store the functions.

'X^2 [ENTER]

```
4:
3:
2:
1:                   'X^2'
```

'F [STO]

```
4:
3:
2:
1:
```

'X+3 [ENTER]

```
4:
3:
2:
1:                   'X+3'
```

'Y [STO]

```
4:
3:
2:
1:
```

Plot both curves to find the intersection points.

'F=Y [ENTER]

```
4:
3:
2:
1:                'F=Y'
```

[EVAL]

```
4:
3:
2:
1:          'X^2=X+3'
```

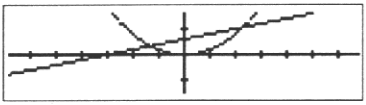
Store the equation and set up the plot parameters. If you have no prior knowledge of the graph of the curves, you can first draw the graph, exit and modify the plot parameters as shown below, and then proceed with a second graph.

[PLOT] ≣STEQ≣
'PPAR [PURGE]
5 ≣*H≣

```
3:
2:
1:
PPAR | RES | AXES |CENTR| *W | *H
```

The rightmost intersection point will become the upper limit. The left-most intersection point is the lower limit. Draw the equation and digitize the rightmost point first, followed by the leftmost point.

≣DRAW≣



[>] . . . [↑] [INS]
[↓] . . . [<] [INS]
[ATTN]

```
3:
2:              (2.30,5.50)
1:             (-1.40,2.00)
STEQ|RCEQ|PMIN|PMAX|INDEP|DRAW
```

Use the Solver to refine the initial guess.

[SOLV] ≣SOLVR≣

```
3:
2:              (2.30,5.50)
1:             (-1.40,2.00)
 X |LEFT=| RT= |    |    |
```

≣X≣
[ ] ≣X≣

```
X: -1.30
Sign Reversal
1:                    -1.30
 X |LEFT=| RT= |    |    |
```

Repeat the process for the upper limit.

[SWAP]
[X]
[ ] [X]

```
X: 2.30
Sign Reversal
1:                2.30
  X  |LEFT=| RT= |     |     |
```

The limits are in the correct order for integration but the variable is missing. Manipulate the stack to put it in place.

'X [ENTER]
3 [STACK] [ROLLD]

```
3:                 'X'
2:               -1.30
1:                2.30
ROLLD|PICK |DUPN |DROPN|DEPTH|→LIST
```

Now convert the 3 elements to a list.

3 [LIST] [→LIST]

```
3:
2:
1:      { X -1.30 2.30 }
→LIST|LIST→| PUT | GET |PUTI |GETI
```

Recall the integrand.

[USER] [AREA]

```
3:
2:      { X -1.30 2.30 }
1:          'ABS(F-Y)'
  X  |PPAR | EQ  |  Y  |  F  |AREA
```

Put them in the necessary order.

[SWAP]

```
3:
2:          'ABS(F-Y)'
1:      { X -1.30 2.30 }
  X  |PPAR | EQ  |  Y  |  F  |AREA
```

Enter the error and integrate.

.005 [ENTER]

```
3:          'ABS(F-Y)'
2:      { X -1.30 2.30 }
1:                0.01
  X  |PPAR | EQ  |  Y  |  F  |AREA
```

[∫]

```
3:
2:                7.81
1:                0.04
 X  PPAR  EQ  Y  F  AREA
```

The area is 7.81.

Purge the variables created in this problem section.

`{AREA F Y EQ X PPAR` [ENTER] [PURGE]

Arc Length

This section demonstrates keystroke and programming examples for computing arc lengths of rectifiable functions. The program ARC created in the second example is used in a later section entitled "Surface Area".

Example: Find the length of the curve

$$F(x) = \frac{(\sqrt{x^2+2})^3}{3}$$

from $x = 0$ to $x = 3$.

The arc length of a function is found by evaluating the integral

$$\int_a^b \sqrt{1+f'(x)^2}\,.$$

First form the integrand. Enter the given function in terms of x.

[CLEAR] [<>]
' ((X^2+2)^(3÷2))÷3 [ENTER]

```
4:
3:
2:
1:     '(X^2+2)^(3/2)/3'
```

Specify the variable of differentiation.

'X [ENTER]

```
4:
3:
2:     '(X^2+2)^(3/2)/3'
1:                  'X'
```

Take the derivative and simplify.

[d/dx]

```
3:
2:
1: '2*X*1.50*(X^2+2)^
   0.50/3'
```

Collect terms.

[ALGEBRA] COLCT

```
3:
2:
1:      '(2+X^2)^0.50*X'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Square the derivative, add one, and take the square root.

[x^2] 1 [+] [√]

```
2:
1: '√(SQ((2+X^2)^0.50*X
   )+1)'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

This is the differential of arc length.

Place the list containing the variable and limits of integration on the stack.

{X 0 3 [ENTER]

```
3:
2: '√(SQ((2+X^2)^0.50*…
1:       { X 0.00 3.00 }
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Specify the accuracy and perform the integration.

.005 [ENTER]
[∫]

```
3:
2:                 12.00
1:                  0.06
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

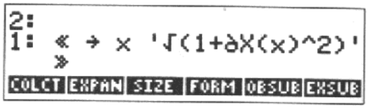
The arc length is 12.00.

Example: Compute the arc length of $f(x)=x^2$ for $x=0$ to $x=2$.

For repeated problems, a simple program facilitates the computation of arc length. The program below differentiates the function with respect to X. This means that functions must be entered in terms of X.

The partial derivative symbol '∂' is obtained by pressing the [d/dx] key.

[CLEAR]
`« → x '√(1+∂X(x)^2`
[ENTER]



Examine this function to see that it is equivalent to the integrand in the previous example.

Store the program in the variable ARC.

`'ARC` [STO]

The program below first stores the error in the variable *ER*, then converts the next three levels of the stack to the list required for integration. The function is then brought to level 1 and operated on by the ARC function. Finally the function is returned to its position and the error is recalled. The integration completes the process.

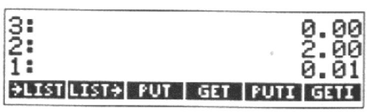
```
« 'ER' STO 3 →LIST
SWAP ARC SWAP ER
∫
```
[ENTER]

Store the program ARCP.

`'ARCP` [STO]

Computing the arc length of any function now only requires placing the correct information on the stack. This program requires the function on level 5, the variable of integration on level 4, the upper limit on level 3, the lower integration limit on level 2, and the error bound on level 1.

`'X^2' 'X' 0 2 .005`
[ENTER]



Compute the arc length.

USER ≣ARCP≣

Purge the program ARCP and variable ER. Program ARC is used in the next problem section.

'ARCP PURGE 'ER PURGE

Surface Area

The function created to compute arc lengths can be extended to computing surface areas.

Example: Compute the surface area of the solid formed by revolving the section of $f(x)=x^2$ between 0 and 1 about the x axis.

In this problem the integrand is expressed in terms of a function of x. The surface area can be computed from

$$S=\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,.$$

The square root factor in the integrand is identical to the ARC function used in the problem section entitled "Arc Length". If you have not already done so, key in the ARC function from the previous section. Enter the integrand using ARC as a function.

[CLEAR] [<>]
'2×π×F×ARC(F [ENTER]

```
4:
3:
2:
1:          '2*π*F*ARC(F)'
```

Enter the function to be integrated.

'X^2 [ENTER]

```
4:
3:
2:          '2*π*F*ARC(F)'
1:                  'X^2'
```

Store the function by the corresponding name appearing in the integrand.

'F [STO]

```
4:
3:
2:
1:          '2*π*F*ARC(F)'
```

Purge the variable of integration to ensure that the name is not in use.

'X [PURGE]

```
4:
3:
2:
1:          '2*π*F*ARC(F)'
```

Enter the variable of integration and the limits.

{X 0 1 [ENTER]

```
4:
3:
2:          '2*π*F*ARC(F)'
1:       { X 0.00 1.00 }
```

Enter the error bound and compute the surface area.

.005 [ENTER]
[∫]

```
4:
3:
2:                   3.81
1:                   0.02
```

The surface area is 3.81.

Purge the variables created in this problem section.

{F ARC [ENTER] [PURGE]

Arc Length of Parametric Equations

It is often necessary to work with equations expressed in terms of a parameter. The coordinates of a particle moving in a plane as a function of time is a common example.

Example: Compute the length of the curve corresponding to the equations

$$x(t)=\frac{t^2}{2} \quad \text{and} \quad y(t)=\frac{(2t+1)^{\frac{3}{2}}}{3}$$

for $t=0$ to $t=4$.

In parametric form the arc length is

$$L=\int_a^b \sqrt{dx^2+dy^2}\ .$$

Enter the integrand in terms of the differentials of x and y. This general relationship can be used for any set of parametric equations with T as the parameter.

[CLEAR] [<>]
'√(SQ(∂T(X))+SQ(∂T(Y
[ENTER]

```
3:
2:
1:  '√(SQ(∂T(X))+SQ(∂T(Y
    )))'
```

Save the parametric arc length in PARC.

'PARC [STO]

```
4:
3:
2:
1:
```

Enter the parametric equations. Store them under the names X and Y as expected by the PARC function.

'T^2÷2 [ENTER]
'(2×T+1)^(3÷2)÷3 [ENTER]

```
4:
3:
2:                'T^2/2'
1:     '(2*T+1)^(3/2)/3'
```

'Y [STO]
'X [STO]

```
4:
3:
2:
1:
```

Now integrate with respect to *T* from 0 to 4.

First recall the integrand.

[USER] PARC

```
2:
1:  '√(SQ(∂T(X))+SQ(∂T(Y
    )))'
  X    Y   PARC
```

Key in the variable of integration and the limits.

{T 0 4 [ENTER]

```
3:
2:  '√(SQ(∂T(X))+SQ(∂T(…
1:        { T 0.00 4.00 }
  X    Y   PARC
```

Enter the desired error bound.

.005 [ENTER]

```
3:  '√(SQ(∂T(X))+SQ(∂T(…
2:        { T 0.00 4.00 }
1:                   0.01
  X    Y   PARC
```

Now perform the integration.

[∫]

```
3:
2:                  12.00
1:                   0.06
  X    Y   PARC
```

The arc length is 12.00.

Program PARC is used in the next section, and *X* and *Y* are replaced by new functions.

Surface Area of Parametric Equations

The function created to compute arc lengths can be extended to compute surface areas. The surface area can be found by computing the integral

$$S = \int_{a}^{b} 2\pi Y \sqrt{dx^2 + dy^2}$$

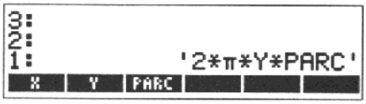
Example: Determine the surface area of the sphere formed by revolving a circle about the x axis.

$$x(t) = 2\cos(t) \qquad y(t) = 2\sin(t)$$

These are the parametric equations for a circle of radius 2.

Note that the integrand includes the parametric arc length as a factor. Use the function defined in the previous section in the integrand. Clear user flag 35 for numeric evaluation of π when it is supplied as a limit to the integration.

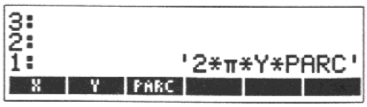
[CLEAR] 35 CF [ENTER]
'2×π×Y×PARC [ENTER]



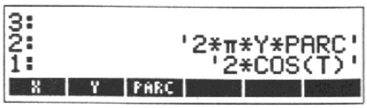
Now enter the X and Y equations.

'2×SIN(T [ENTER]

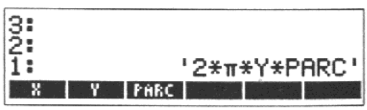
'Y [STO]



'2×COS(T [ENTER]



'X [STO]



Key in the variable and limits of integration. With flag 35 cleared, π is evaluated to its numeric representation. The integration that follows requires a non-symbolic representation. Convert the parameters into a list.

T 0 π [ENTER]
3 [LIST] →LIST

```
3:
2:          '2*π*Y*PARC'
1:       { T 0.00 3.14 }
→LIST LIST→ PUT GET PUTI GETI
```

Key in the error bound and perform the integration.

.005 [ENTER]
[∫]

```
3:
2:                 50.27
1:                  0.25
→LIST LIST→ PUT GET PUTI GETI
```

Note that 50.27 is $4\pi r^2$.

Purge the programs and variables created in this problem section.

{X Y PARC [ENTER] [PURGE]

Volume of Solid of Revolution: Method of Shells

This section demonstrates computation of the volume of a solid of revolution by the method of shells.

The method of shells requires evaluation of the integral

$$\int_a^b 2\pi x F(x)dx \ .$$

Example: Find the volume of the solid formed by revolving the curve

$$F(x)=e^{-x^2}$$

from $x=0$ to $x=3$ about the Y axis. Consider the behavior of the integral as the region of integration is extended.

Form an algebraic expression for the integrand including a general function $F(x)$.

[CLEAR] [<>]
'2×π×X×F [ENTER]

```
4:
3:
2:
1:                '2*π*X*F'
```

Store the integrand.

'SHEL [STO]

```
4:
3:
2:
1:
```

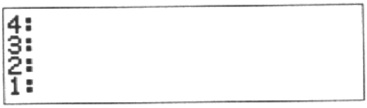
Now enter the function. This must be a function of X as specified in the volume integrand.

'EXP(-X^2 [ENTER]

```
4:
3:
2:
1:              'EXP(-X^2)'
```

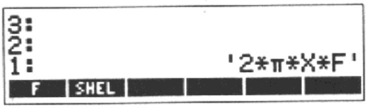
Store the function by the name used in the SHEL program.

'F [STO]



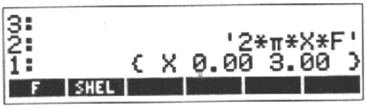
Recall the expression to be integrated.

[USER] SHEL



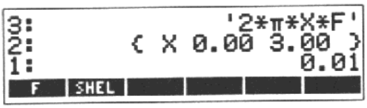
Place the variable of integration and the limits on the stack.

{X 0 3 [ENTER]



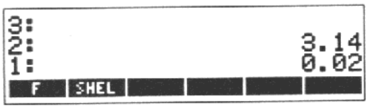
Specify the error bound of the integration.

.005 [ENTER]



Now integrate the function.

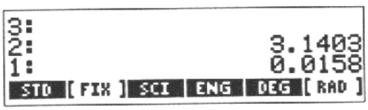
[∫]



The result corresponds to π within the error specified.

Reset the display to show four digits.

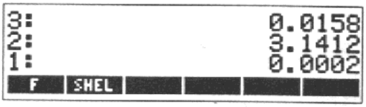
[MODE] 4 FIX



As expected, the accuracy is limited by the specification of two digits.

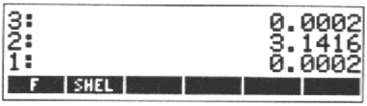
Perform the integration again, increasing the accuracy to produce four digits to the right of the decimal.

USER SHEL
{X 0 3 ENTER
.00005 ENTER
∫



The desired accuracy was not achieved. By extending the region of integration, it may be possible to generate more digits of accuracy.

SHEL
{X 0 4 ENTER
.00005 ENTER
∫



This is indeed π to four digits. This process does not prove that the integral, taken to infinity, converges to π. That proof requires an explicit solution to the integral. The curve that was specified is, of course, the "bell curve" used frequently in statistical analysis.

Purge the programs and variables used in the last two sections.

{SHEL F ENTER PURGE

Volume of Solids of Revolution : Method of Disks.

This problem section computes volume of solids of revolution by the method of disks.

The method of disks requires evaluation of the integral

$$\int_a^b \pi f(x)^2 dx .$$

In general, for a given integral, the smaller the error bound the longer the integration will take. The appropriate choice of error bound depends on the problem being solved, but the method to reach a solution remains constant.

Example: Compute the volume of the solid formed by revolving the function $f(x)=x^2$ from 0 to 1 about the x axis.

Key in the first program for the general form of the integrand.

[CLEAR] [<>]
« → x 'π×X^2 [ENTER]

```
4:
3:
2:
1:        « → x 'π*x^2' »
```

Store the program in the variable DSK.

'DSK [STO]

```
4:
3:
2:
1:
```

Key in the second program. This program puts the function and integration parameters in the appropriate form on the stack and calls DSK for the general form of the integrand. It then performs the volume computation.

« 'ER' STO 3.00 →LIST
SWAP DSK SWAP ER ∫
[ENTER]

```
2:
1: « 'ER' STO 3.0000
   →LIST SWAP DSK SWAP
   ER ∫ »
```

Store the second program by the name DSKP.

'DSKP [STO]

```
4:
3:
2:
1:
```

Now enter the function and integration data.

'X^2''X' 0 1 .005 [ENTER]

```
4:                'X'
3:             0.0000
2:             1.0000
1:             0.0050
```

Execute the program.

[USER] [DSKP]

```
3:
2:             0.6283
1:             0.0031
 ER  DSKP  DSK
```

The computed volume is .6283. The explicit solution to the integral is $\pi/5$.

For greater accuracy, increase the error bound as appropriate.

Purge the programs and variables created in this problem section.

{DSK DSKP ER [ENTER] [PURGE]

Step-by-Step Examples for Your HP-28C

Calculus contains a variety of examples and solutions to show how you can solve your technical problems more easily.

- Function Operations
  Definition, Composition, Analysis, Angle Between Lines and Functions
- Differential Calculus
  Maximization/Minimization, Differentiation and Tangent Lines, Implicit Function Theorem
- Integral Calculus
  Integration and Free Falling Bodies, Double Integrals and Area Between Two Curves, Arc Length and Surface Area, Volume of Solids of Revolution

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