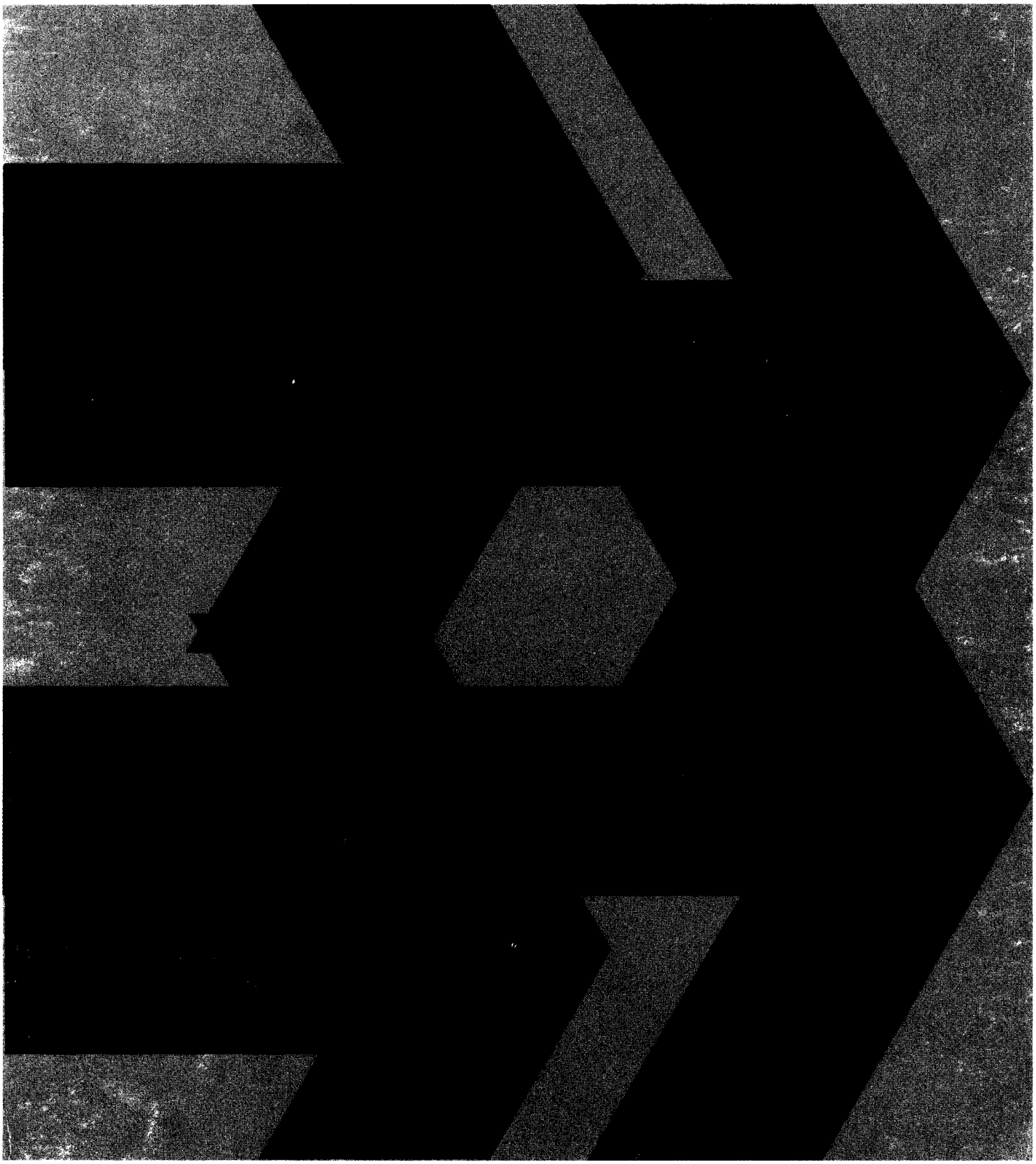
HEWLETT PACKARD COMPUTER CURRICULUM

COLLEGE AND UNIVERSITY SERIES

ELECTRIC AND MAGNETIC FIELDS

by John R. Merrill



HEWLETT PACKARD COMPUTER CURRICULUM

COLLEGE AND UNIVERSITY SERIES

ELECTRIC AND MAGNETIC FIELDS

by **JOHN R. MERRILL**
Dartmouth College
Hanover, New Hampshire

Exercises and Solutions Prepared With
Gregory P. Hughes, Dartmouth College

HEWLETT-PACKARD COMPANY
11000 WOLFE ROAD
CUPERTINO, CALIFORNIA 95014

HP 5951-5667

Printed: April 1974

The Hewlett-Packard Computer Curriculum Series represents the published results of a Curriculum Development project sponsored by the Data Systems Division of Hewlett-Packard Company. This project is under the directorship of Jean H. Danver.

This material is designed to be used with any Hewlett-Packard system with the BASIC programming language such as the 9830A, Educational BASIC, and the 2000 and 3000 series systems.

The Hewlett-Packard Company makes no warranty, expressed or implied, and assumes no responsibility in connection with the operation of the program material attached hereto.

PREFACE

This unit shows you ways in which the computer can extend your knowledge and understanding of introductory electricity and magnetism. The unit presents numerical solutions for field lines and equipotentials. The solutions are easy to understand and yet include solutions to many problems which can't be handled by analytic methods. The emphasis in the unit will be on ways the computer can extend the electricity and magnetism you understand. The unit assumes some knowledge of introductory e and m, specifically the fields for simple charge distributions.

Your instructor can assign chapters and exercises out of this unit in much the same way as out of a textbook. Lectures on the material may be presented, but you should be able to understand the material without lectures. The harder exercises can be used as the basis of projects if your instructor wishes.

Each chapter in the unit starts with a brief discussion of the physics discussed in the chapter and then moves on to an explanation of the numerical procedure used with the computer. Exercises follow with one exercise completely worked out in the text so that you can see what a sample solution looks like. This sample exercise is sometimes a problem for which the answer is already known and therefore provides an extra check on the computer method. Some exercises are marked with asterisks. One asterisk means the exercise is fairly demanding while two asterisks indicates a very challenging problem. Numerical methods will not be discussed much in the text; there is a section in the appendix which discusses the half-step integration used.

Throughout the booklet normalized or natural units will be used. The units have the advantage that the numbers being calculated stay near 1; therefore rarely getting too small or too large for the computer. Natural units are often used in e and m; the units have the effect of setting $1/(4\pi\epsilon_0)$ and $\mu_0/(4\pi)$ all equal to one.

Often graphical output is useful. If you have a plotter or a CRT terminal available, fine. If not, terminal plotting (using the teletype to plot a graph) can be helpful. The appendix gives a program to convert printing programs (programs which type out lists of numbers) to programs which plot on the terminal. This terminal plotting will be used several times in the unit to give you several examples.

John Merrill
Tallahassee, Florida

This unit was written while the author was on the faculty of the Department of Physics and Astronomy at Dartmouth College, Hanover, New Hampshire. John Merrill has used computers to support his classwork and research since joining the Dartmouth faculty in 1966. During the period of 1970-72 he was part of a team of professors who developed computer uses for the classroom in engineering, mathematics and physics under the auspices of NSF funded project COEXIST.

John has written many journal acticles on specific uses of computers in physics teaching, particularly for the two main physics teaching journals, *The Physics Teacher* and *The American Journal of Physics.* He has written several booklets on computer use in physics teaching including another unit in the Hewlett-Packard series, *Quantum Mechanics.* John is also the

author of the book, *Computers in Physics,* soon to be published by Houghton-Miffin Company. He is now the Director of the Center for Educational Design at Florida State University.

Special credits go to Gregory Hughes, a recent PhD in physics from Dartmouth, who helped develop the exercises and Christine Doerr, who copyedited the manuscript.

TABLE OF CONTENTS

1 CHAPTER ONE: ELECTROSTATICS
1 Introduction
1 The Electrostatic Field, $\vec{E}(\vec{r})$
2 The Electrostatic Potential, $V(\vec{r})$
3 Visualizing Electrostatic Fields
7 Field Lines and Equipotentials with the Computer
16 Conclusion
17 Chapter One Exercises

21 CHAPTER TWO: MAGNETOSTATIC FIELDS
21 Introduction
21 Magnetic Fields for Various Current Distributions
22 Mapping Field Lines
24 Flux Lines for a Set of Long Straight Wires
27 The Density of Flux Lines
27 Field Lines for Complicated Current Distributions
30 Conclusion
31 Chapter Two Exercises

35 CHAPTER THREE: THE MOTION OF CHARGED PARTICLES IN ELECTROSTATIC AND MAGNETOSTATIC FIELDS
35 Introduction
35 An Algorithmic Solution to Newton's Second Law
36 The Motion of a Charged Particle in an Electrostatic Field, $\vec{E}$
39 The Motion of Charged Particles in Combined Electrostatic and Magnetostatic Fields
41 Conclusion
41 Chapter Three Exercises

45 CHAPTER FOUR: GAUSS'S, AMPERE'S, AND FARADAY'S LAWS FOR STATIC FIELDS
45 Introduction
46 Pictures of Vector Fields
52 Surface and Line Integrals with the Computer
54 Conclusion
55 Chapter Four Exercises

59 CHAPTER FIVE: THE LAPLACE AND POISSON EQUATIONS
59 Introduction
59 Difference Equations
60 The Difference Equations for Laplace's and Poisson's Equations
64 Conclusion
64 Chapter Five Exercises

69 APPENDIX A: TERMINAL PLOTTING

73 APPENDIX B: THE HALF-STEP ITERATIVE INTEGRATION

75 ANSWERS TO SELECTED EXERCISES

CHAPTER ONE: ELECTROSTATICS

INTRODUCTION

Electrostatics introduces the basic concept of fields, a concept used widely in electricity and magnetism (e & m) as well as in other areas of physics. The purpose of this chapter is to deepen your understanding of fields by using the computer to map electrostatic fields in various ways. You will map electrostatic patterns both by using the electrostatic potential and by using the electrostatic field itself.

THE ELECTROSTATIC FIELD, $\vec{E}(\vec{r})$

A field is a way to visualize a vector whose magnitude and direction vary as you move around space. Consider the Coulomb force, $\vec{F}_{21}$, on a charge, q_2, due to a charge, q_1, a distance, $\vec{r}$, away,

$$\vec{F}_{21} = k_0 \frac{q_1 q_2}{r^2} \left(\frac{\vec{r}}{r} \right) \qquad (1)$$

where $r = |\vec{r}|$ and k_0 defines the units. $k_0 = 1/4\pi\epsilon_0 = 9\times10^9$ in mks units; $k_0 = 1$ in Gaussian (cgs) units. This force means that q_1 reaches out over the distance, $\vec{r}$, in order to create the force felt by q_2. Originally, the electrostatic field, $\vec{E}$, was introduced into physics so that q_2 would be interacting with something (namely, the $\vec{E}$ field) right where q_2 was. The idea is that q_1 produces an $\vec{E}$ field everywhere, and that q_2 interacts with the $\vec{E}$ field at q_2's position.

The $\vec{E}$ field turns out to be a very convenient way to handle electrostatics problems because you can separate the effects of q_1, the so-called source charge, from the effects of the field upon q_2. You can separate each problem into a part dealing with what field is produced by the source charges and another separate part dealing with the behavior of other charges placed in that field.

The electrostatic field at some point, $\vec{r} = (x,y,z)$, is defined as the force per unit charge on a test charge, q_2, placed at $\vec{r}$.

$$\vec{E}(\vec{r}) = \frac{\vec{F}(\vec{r})}{q_2} \qquad (2)$$

where $\vec{F}(\vec{r})$ is the force felt by q_2 at the point $\vec{r}$.

Equations for the electrostatic field, $\vec{E}$, for simple kinds of source charges are derived in regular textbooks. A short table of the fields looks like the following:

Charge Distribution	$\vec{E}$ Field
Single Point Charge	$k_0\, q/r^2\ (\vec{r}/r)$
Single Line Charge	$k_0\, \lambda/r\ (\vec{r}/r)$

λ = the line charge density

The total electrostatic field due to some set of source charges is just the (vector) sum of the fields due to each source charge individually.

Practice Exercise: Show that the force experienced by a test charge, q_2, placed at the point $\vec{r}$ is the Coulomb force when the source charge is a point charge.

THE ELECTROSTATIC POTENTIAL, $V(\vec{r})$

Another useful concept in electrostatics is the electrostatic potential, $V(\vec{r})$. You can define or introduce the potential in several different ways:

1. When the source charge distribution does not extend to ∞, the potential can be defined as the work per unit charge done to bring a test charge, q_2, from infinite separation to the point, $\vec{r}$.

   The work done by a force $\vec{F}$ between $\vec{r}_1$ and $\vec{r}_2$ is defined as $\int_{r_1}^{r_2} \vec{F} \cdot d\vec{\ell}$, so the potential is

$$V(\vec{r}) \equiv q \int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{\ell} = -q \int_{\vec{r}}^{\infty} \vec{E} \cdot d\vec{\ell} \qquad (3)$$

   Even when the source charge distribution does extend to infinity, some convenient position, $\vec{r}_0$, can be chosen, and potentials can be measured from that point.

$$V(\vec{r}) \equiv q \int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{\ell} \qquad (4)$$

2. The potential can also be introduced from its other relationship to the field, $\vec{E}$, namely

$$\vec{E}(\vec{r}) \equiv -\text{grad}\,(V(\vec{r})) \qquad (5)$$

   The gradient, called *grad,* is just a vector derivative.

These two definitions of potential are essentially equivalent. We will not do much with the second definition for the potential, $V(\vec{r})$, but it is the more general of the two forms. This second definition is also the one used most in upper levels of e & m study. One very useful result of this second definition is that the electrostatic field, $\vec{E}(\vec{r})$, at the point, $\vec{r}$, points in the direction of steepest decrease of the electrostatic potential, $V(\vec{r})$, at that point $\vec{r}$. The force on a test charge at $\vec{r}$ always point directly down the potential hill.

The first definition points out most clearly that, unlike the $\vec{E}$ field, which is a vector function of position, the electrostatic potential, $V(\vec{r})$, is a scalar function of position. $\vec{E}(\vec{r})$ has both magnitude and direction at each point, $\vec{r}$, whereas the potential, $V(\vec{r})$, has only a value at each point, $\vec{r}$. The fact that V is a scalar makes it very easy to use, which is one of the reasons the potential appears so often.

Again, textbooks derive the potentials for various simple charge distributions. A table follows:

Charge Distribution	Potential
Point Charge, q	$k_0\ q/r$
Line Charge, λ	$-2\ k_0\ \lambda \ln(r/r_0)$

λ = the line charge density

Practice Exercise: Show that the potential due to a point charge follows from the integral definition of potential and the earlier equation for the electrostatic field due to a point charge.

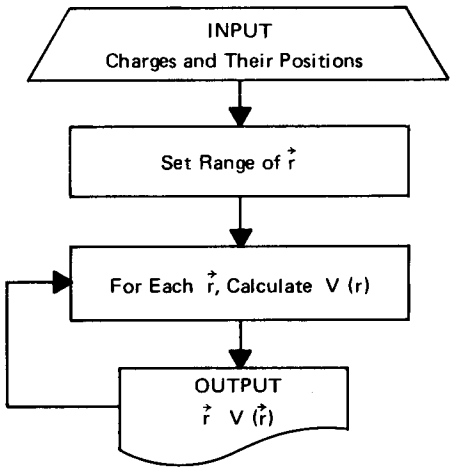
Analytical methods which solve electrostatics problems are generally limited to those which can be integrated easily. Numerical, computer-based solutions are not so limited. For a general charge distribution, you can break the distribution up into pieces (which can be treated as point charges or line charges or whatever) and then just add up the fields or potentials due to each piece. The computer-based methods are very general and yet are based only on the simple equations for fields and potentials (those quoted above).

VISUALIZING ELECTROSTATIC FIELDS

There are several ways to visualize electrostatic fields, $\vec{E}$. The first way uses the electrostatic potential, V. The starting point is a plot of the potential everywhere in space. (Everywhere means at lots of points throughout some region of interest.) You then connect all the points having the same value of potential. The resulting surfaces in three dimensions (or contours in two dimensions) are called equipotentials. No work is done on or by a test charge moved around on any equipotential surface. The forces on any test charge are always perpendicular to equipotential surfaces. Because the electrostatic field is the force per unit charge on a test charge, q_2, the $\vec{E}$ field is always perpendicular to equipotential surfaces. Thus you can visualize what happens to a test charge (in some field set up by given source charges) by finding the equipotentials in the field. You can also (crudely) find the electrostatic field, $\vec{E}$, at any point, $\vec{r}$, by discovering in which direction away from $\vec{r}$ the potential decreases fastest. The greater the rate of change of potential, the stronger is the electric field, $\vec{E}$.

All you need to be able to do to use this method is to be able to calculate the potential everywhere due to any given source charge distribution. However, the equation for an equipotential, is difficult to derive even for fields as simple as that due to three point charges. With the computer you

just add up the potentials due to whatever set of source charges you have: for three point charges, q_1, q_2, q_3 at $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ and for any point, $\vec{r}$, you sum $k_0\ (q_1/|\vec{r}-\vec{r}_1| + q_2/|\vec{r}-\vec{r}_2| + q_3/|\vec{r}-\vec{r}_3|)$. For N charges, you add up N such terms. If the source distribution is some mixture of point charges, line charges and whatever else, you must remember to put the right form of the potential in each term. The method is easy and very general. A block diagram of this strategy for potentials is shown below:



A second method is occasionally used to visualize electrostatic fields. In this method, the vectors $\vec{E}(\vec{r})$ are calculated at lots of points over the whole region of interest. Arrows representing the various fields, $\vec{E}(\vec{r})$, at the various points will be short where $|\vec{E}|$ is small and long where $|\vec{E}|$ is large. You can then visualize the $\vec{E}$ field in terms of constant $|\vec{E}|$ surfaces (or contours in two dimensions). Although this is a valid way to visualize fields, it is not very common. This method is illustrated by a problem at the end of this chapter.

The third and most common way to visualize electrostatic field patterns uses field lines. One of the main points of this chapter is to allow you to use simple computer programs to generate field line patterns for numbers of different source charge distributions.

The two-part definition of a field line is deceptively simple:

1. At every point, $\vec{r}$, the electrostatic field, $\vec{E}$, is tangent to (that is, is locally parallel to) the field line.

2. The number of field lines passing through a (unit, perpendicular) cross sectional area at $\vec{r}$ is proportional to $|\vec{E}(\vec{r})|$.

This second part of the definition simply means that the field is strong where field lines bunch together and weak where lines are spread apart.

Notice a few points about field lines:

a. $|\vec{E}|$ is usually not constant along a field line. Any line usually goes through regions where $\vec{E}$ is weak and regions where $\vec{E}$ is strong.

b. Field lines start on positive charges and end on negative charges. If the total net charge of a charge distribution is not zero (that is, if there is more positive charge than negative, or vice versa) then some field lines may begin or end at infinity, since the lines will keep on searching for some charge on which to terminate.

c. The number of lines that start on a +2 charge is twice the number that start on a +1 charge. Similarly, twice as many lines will end on a -2 charge as end on a -1 charge. You can choose arbitrarily how many lines start on a +1 charge (note that the second part of the field line definition says "proportional to"), but having chosen that number, all else follows.

At first sight the definition of field line seems very simple: after all, there are only two defining characteristics. At second glance, the definition seems to make field lines very complicated. Analytical solutions for field lines can only be performed for very simple source charge distributions, and even then the expressions look very complicated. Using a computer the field lines can be mapped using just the definition and a very short program.

Consider the following way to trace out a field line. Suppose you know some point, $\vec{r} = (x,y,z)$, on a field line. (Any point lies on some field line, so you define the field line you are going to trace by choosing the first point.) Suppose further that you want to take a step of size Δs along that field line. You calculate the field at r, $\vec{E}(\vec{r})$, and then use the first defining property (the line is parallel to $\vec{E}$ at $\vec{r}$) to calculate

$$\Delta x = \Delta s \, \vec{E}_x/|\vec{E}|$$

$$\Delta y = \Delta s \, E_y/|\vec{E}|$$

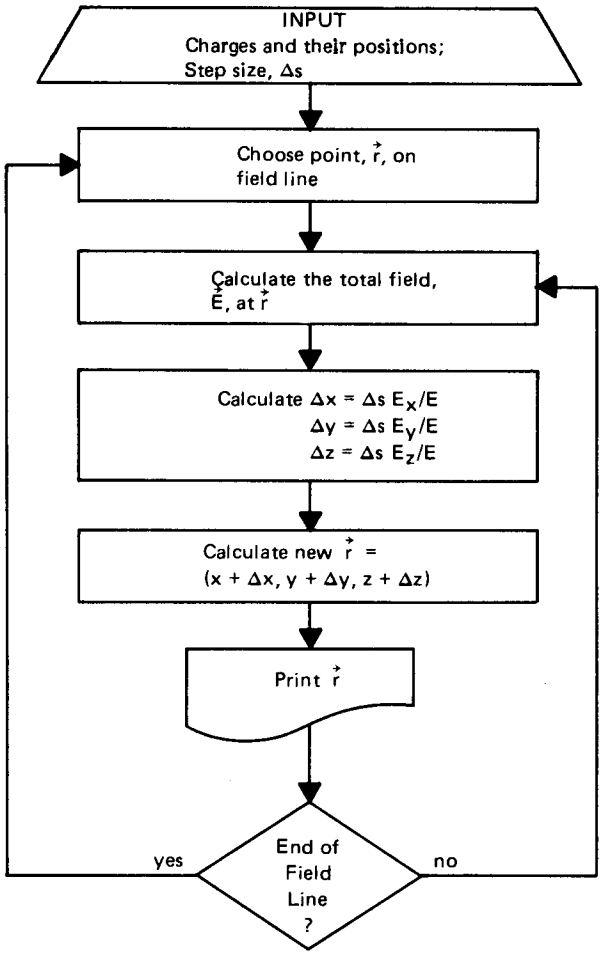
$$\Delta z = \Delta s \, E_z/|\vec{E}| \qquad (6)$$

These equations say that $\vec{\Delta s} = \vec{\Delta x} + \vec{\Delta y} + \vec{\Delta z}$ is parallel to $\vec{E} = \vec{E}_x + \vec{E}_y + \vec{E}_z$ which is equivalent to the first defining property.

Practice Exercise: When $\vec{E}_z$=0, show that the triangle Δx, Δy, Δs is similar to the triangle $\vec{E}_x$, $\vec{E}_y$, $\vec{E}$.

The next point along the field line is then $(x+\Delta x, y+\Delta y, z+\Delta z)$, and you can repeat the process at this new point. You walk along the field line step-by-step. Such a method of solution is called "algorithmic" (because the solution is an algorithm or procedure) or "iterative" (because you iterate or repeat the same steps over and over). Notice that the procedure breaks down if $|\vec{E}| = 0$ at any point on the field line. Such methods are

easy on computers. A block diagram of this procedure is shown below:



Notice that you could perform the calculation yourself if you had the patience; there is nothing magic about the method at all.

What about the second part of the field line definition? How do you ensure that your pattern will obey that property, too? The answer lies in how you choose to start field lines. Close enough to any charge, the field lines will be radial since all the other charges are so far away their effects are negligible. Close to each positive charge you can choose the number of lines (q times the number you choose to start on a +1 charge) to start radially and at equal angles. Once the lines are started correctly, the whole pattern will remain correct. More lines will bend around so that they go through regions of strong field, and fewer lines will bend around so as to go through weak field regions. This simple method produces results automatically.

The same sort of procedure can be used to trace out equipotentials. Since equipotentials are always perpendicular to field lines, we move perpendicularly to the field, $\vec{E}$, at each point (rather than parallel to $\vec{E}$). In two dimensions (or symmetrical three-dimensional systems such as those we consider below) this procedure is particularly easy. The steps Δx and Δy, parallel to x and y, are given by $\Delta x = -\Delta s\, E_y/|\vec{E}|$ and $\Delta y = \Delta s\, E_x/|\vec{E}|$.

FIELD LINES AND EQUIPOTENTIALS WITH THE COMPUTER

Let us do an example of this procedure which is similar to the problems at the end of the chapter. We will choose a relatively simple example: two opposite charges (+1 and -1) at (+.5, 0) and (-.5, 0) respectively. Far away from the charges ($|\vec{r}|>>1$) this pattern reduces to a dipole pattern. (This is actually one of the few problems which can be solved analytically. Hence this example can also serve to verify the numerical method; this verification is discussed in a problem.)

Since the pattern is symmetric under rotations around the x-axis, we can limit our discussion to the x-y plane. That is to say, the z direction is just like the y direction, so the full three-dimensional picture is just the one we will produce rotated around the x-axis. Two-dimensional pictures are easier to put on paper; the method, of course, works just as well in three dimensions. Most of the physical intuition can be derived from two-dimensional pictures.

Example 1. Find the potential for the above charge distribution. Consider $-5<x<5$ and $-5<y<5$.

The potential is just $V = k_0(+1/[\text{distance from } (+.5,0)]) + k_0(-1/[\text{distance from } (-.5, 0)])$. One program which calculates and prints-out the values of the potential is listed below:

```
POTENT

100  PRINT "(X,Y) OF LEFT-BOTTOM & RIGHT-TOP?"   Specify Region
110  INPUT X8,Y8,X9,Y9                           of interest
120  LET K0=1 - Set units
130  LET Q=+1 - Magnitude of changes
140  LET N0=9 - # of points across x and y
150  LET X7=(X9-X8)/N0 - Δx
160  LET Y7=(Y9-Y8)/N0 - Δy
170  FOR J=N0 TO 0 STEP -1 - Step down y
180  LET Y0=Y8+J*Y7 - Each y in turn
190  FOR I=0 TO N0 - Step across x
200  LET X0=X8+I*X7 - Each x in turn
210  LET R=SQR((X0-.5)*(X0-.5)+Y0*Y0) - Distance from (+.5,0)
220  LET R1=SQR((X0+.5)*(X0+.5)+Y0*Y0) - Distance from (-.5,0)
230  LET V=K0*Q/R-K0*Q/R1 - Potential
240  PRINT V, - Print potential
250  NEXT I
260  PRINT
270  NEXT J
280  PRINT
290  GOTO 100 - Return for new region
300  END
```

Practice Exercise: Modify the program so that it computes the potential for two equal, positive charges at (+.5, 0), (-.5, 0).

A RUN of this program produces the following output. Notice that each value of y produces two lines of printing; the second line gives the values another page width to the right of the first line.

```
RUN
POTENT

(X,Y) OF LEFT-BOTTOM & RIGHT-TOP?
?-.5,-.5,.5,.5
-1.10557        -.971852        -.746108        -.464101        -.156889
 .156889         .4641           .746108         .971852         1.10557

-1.63942        -1.44181        -1.08265        -.656706        -.21868
 .21868          .656706         1.08265         1.44181         1.63942

-2.63648        -2.26873        -1.60032        -.920048        -.298028
 .298028         .920048         1.60032         2.26873         2.63648

-5.01361        -3.88657        -2.34283        -1.22807        -.382654
 .382654         1.22807         2.34283         3.88657         5.01361

-17.0015        -6.92704        -3.08319        -1.46436        -.441558
 .441558         1.46436         3.08319         6.92703         17.0015

-17.0015        -6.92704        -3.08319        -1.46436        -.441558
 .441558         1.46436         3.08319         6.92703         17.0015

-5.01361        -3.88657        -2.34283        -1.22807        -.382653
 .382653         1.22807         2.34283         3.88657         5.01361

-2.63648        -2.26873        -1.60032        -.920048        -.298028
 .298028         .920048         1.60032         2.26873         2.63648

-1.63942        -1.44181        -1.08265        -.656706        -.21868
 .21868          .656706         1.08265         1.44181         1.63942

-1.10557        -.971852        -.746108        -.464101        -.156889
 .156889         .4641           .746108         .971852         1.10557


(X,Y) OF LEFT-BOTTOM & RIGHT-TOP?
?0,0,1,1
 0               7.87228E-02     .152839         .218173         .271447
 .310716         .335603         .34721          .3477           .339727

 0               .10363          .200662         .285002         .351774
 .398157         .423933         .431343         .424242         .406995

 0               .138997         .268648         .379908         .465109
 .519762         .543989         .542308         .521774         .489679

 0               .189937         .367195         .518171         .629794
 .693829         .711005         .690767         .646572         .590792

 0               .263806         .512489         .725627         .87843
 .952723         .950207         .892262         .806114         .712763

 0               .369911         .728777         1.04792         1.27458
 1.3595          1.30575         1.16882         1.00755         .85562

 0               .515818         1.04749         1.56911         1.96072
 2.05579         1.85912         1.5474          1.25226         1.01331

 0               .694781         1.48774         2.44052         3.33496
 3.43859         2.758           2.04009         1.52167         1.16815

 0               .862522         1.974           3.80283         6.99827
 7.10768         4.13902         2.56285         1.75478         1.28753

 0               .935065         2.21538         4.8             16.9412
 17.0526         5.14286         2.81739         1.85143         1.33333

(X,Y) OF LEFT-BOTTOM & RIGHT-TOP?
?
```

The largest positive numbers for the potential lie near the positive charge, the largest negative values of potential lie near the negative charge.

Sometimes plotting the potential helps you to see the pattern. You can plot characters on the terminal which represent the size of the potential at each (x,y) point on the plot. One way to produce such a plot involves several changes (shown below) in the potential program, POTENT, listed

on page 8. The characters represent the size of the potential; the position of each character is the point (x,y). Terminal plotting is sometimes crude since on most terminals the characters can only appear at definite places on the paper.

```
GET-POTENT

10 DIM A$(36)
20 LET A$="0123456789ABCDEFGHIJKLMNOPQRSTUVWXYZ"
112 PRINT "MIN. & MAX. V?"
114 INPUT V8,V9
140 LET N0=30
160 LET Y7=10*(Y9-Y8)/(6*N0)
170 FOR J=INT(.6*N0+.5) TO 0 STEP -1
240 LET I9=1+INT(35*(V-V8)/(V9-V8)+.5)
241 IF I9>=1 THEN 243
242 LET I9=1
243 IF I9<=36 THEN 245
244 LET I9=36
245 PRINT A$(I9,I9);
```

A RUN of this modified program looks like:

```
RUN
POTENT

(X,Y) OF LEFT-BOTTOM & RIGHT-TOP?
?-.55,-.05,-.45,.05
MIN & MAX V?
?-100,0
UUUUUUTTTTTTSSSSSSSTTTTTTUUUUUU
UUUTTTTTSSSSSSRRRSSSSSSTTTTUUUU
UUTTTTSSSRRRRQQQQQRRRRSSSTTTTUU
UTTTSSSRRQQQPPPPPPPQQQRRSSSTTTU
TTTSSRRQQPPONNNNNNNOPPQQRRSSTTT
TTSSRRQPOONMLKKKKKLMNOOPQRRSSTT
TSSRRQPONMKJHGFEFGHJLMNOPQRRSST
TSSRQPONMKIFC85458CFIKMNOPQRSST
SSRRQPOMLIFB5000005BFILNOPQRRSS
SSRRQPOMKIE9000Z0009EIKMOPQRRSS
SSRRQPOMLIFB5000005BFILNOPQRRSS
TSSRQPONMKIFC85458CFIKMNOPQRSST
TSSRRQPONMKJHGFEFGHJLMNOPQRRSST
TTSSRRQPOONMLKKKKKLMNOOPQRRSSTT
TTTSSRRQQPPONNNNNNNOPPQQRRSSTTT
UTTTSSSRRQQQPPPPPPPQQQRRSSSTTTU
UUTTTTSSSRRRRQQQQQRRRRSSSTTTTUU
UUUTTTTTSSSSSSRRRSSSSSSTTTTUUUU
UUUUUUTTTTTTSSSSSSSTTTTTTUUUUUU

(X,Y) OF LEFT-BOTTOM & RIGHT-TOP?
?-.5,-.5,.5,.5
MIN & MAX V?
?-25,25
HHHHHHHHHHHHHHHIIIIIIIIIIIIIIII
HHHHHHHHHHHHHHHIIIIIIIIIIIIIIII
GGGGHHHHHHHHHHHIIIIIIIIIIIIJJJJ
GGGGGGHHHHHHHHHIIIIIIIIIIIJJJJJ
GGGGGGGGHHHHHHHIIIIIIIIJJJJJJJJ
FFFFGGGGGHHHHHHIIIIIIIJJJJJKKKK
EEEFFFGGGGHHHHHIIIIIIJJJJKKKLLL
CCDEEFFGGGHHHHHIIIIIIJJJKKLLMNN
67ACDEFFGGGHHHHIIIIIJJJKKLMNPST
Z08BDEFFGGGHHHHIIIIIJJJKKLMORZZ
67ACDEFFGGGHHHHIIIIIJJJKKLMNPST
CCDEEFFGGGHHHHHIIIIIIJJJKKLLMNN
EEEFFFGGGGHHHHHIIIIIIJJJJKKKLLL
FFFFGGGGGHHHHHHIIIIIIIJJJJJKKKK
GGGGGGGGHHHHHHHIIIIIIIIJJJJJJJJ
GGGGGGHHHHHHHHHIIIIIIIIIIIJJJJJ
GGGGHHHHHHHHHHHIIIIIIIIIIIIJJJJ
HHHHHHHHHHHHHHHIIIIIIIIIIIIIIII
HHHHHHHHHHHHHHHIIIIIIIIIIIIIIII

(X,Y) OF LEFT-BOTTOM & RIGHT-TOP?
?
```

(The character Ø represents V< = the minimum potential; the character Z represents V> = the maximum potential.)

Example 2. Sketch three equipotentials (approximately) on a plot of potentials.

Using the terminal plots you can connect the points having approximately equal values of potential. Points having the same character are connected (but not Ø or Z); sometimes it is necessary to interpolate between characters (for example to find where 6 would come between printed 5 and 7 characters). The breadth of a region having a given character is broadest when the potential is changing slowly.

Practice Exercise: On the second terminal plot, find the boundary between the I and J equipotentials. What is the shape of this equipotential off the plot?

Practice Exercise: On the first terminal plot, find the K equipotential.

Practice Exercise: Find the contour on which the value of the potential is actually zero on the first terminal plot. Explain its behavior. (It is possible to have the computer hunt through and interpolate between points to find points on a given equipotential.)

Example 3. Sketch three field lines (approximately) on a plot of potentials.

You can find field lines by moving in the direction of the fastest decrease in potential at each point. Using terminal plots the field lines are crude, but you can still get a feeling for this behavior.

Practice Exercise: Starting near the lower left corner of the first terminal plot, follow a field line inward.

Practice Exercise: On the second terminal plot, follow a field line which leaves the positive charge at 135° to the x-axis.

Example 4. Using an algorithmic method, compute three field lines for this charge distribution.

You can use a program which implements the algorithmic strategy discussed above. One program which does this is the following:

```
EMAP

100  LET N0=2 - # of charges
110  FOR J=1 TO N0
120  READ X[J],Y[J],Q[J]          Positions and
130  NEXT J                       Values of Charges
140  DATA .5,0,1,-.5,0,-1
150  LET K0=9.E+09 - Units
160  LET D=.05 - Step size, Δs
170  PRINT "(X,Y) STARTING PT. ON FIELD LINE?"
180  INPUT X5,Y5
190  LET X0=X5 - x along line
200  LET Y0=Y5 - y along line
210  LET X1=0 - Δx
220  LET Y1=0 - Δy
230  LET S0=.5 - Print out every S0 along line
240  LET S=0 - Distance S along line
250  PRINT "X","Y"
260  LET E1=0
270  LET E2=0
280  FOR J=1 TO N0 - Add up fields for each charge
290  LET R3=(X0+X1/2-X[J])*(X0+X1/2-X[J])      |r|
300  LET R3=R3+(Y0+Y1/2-Y[J])*(Y0+Y1/2-Y[J])
310  LET R3=R3↑1.5 - |r|^3
320  LET E=K0*Q[J]/R3 - |E||r|
330  LET E1=E1+(X0+X1/2-X[J])*E - E_x
340  LET E2=E2+(Y0+Y1/2-Y[J])*E - E_y
350  NEXT J
360  LET E0=SQR(E1*E1+E2*E2) - |E|
370  LET X1=D*E1/E0 - Δx
380  LET X0=X0+X1 - New x
390  LET Y1=D*E2/E0 - Δy
400  LET Y0=Y0+Y1 - New y
410  LET S=S+D
420  IF S<S0 THEN 450             Print-out
430  LET S=0                      Lines
440  PRINT X0,Y0
450  FOR J=1 TO N0
460  IF ABS(X0-X[J])+ABS(Y0-Y[J])<.9*D THEN 490   Tests to end
470  NEXT J                                       a line
480  IF ABS(X0)+ABS(Y0)<10 THEN 260 - Far off page
490  PRINT X0,Y0 - Print out last point
500  PRINT
510  GOTO 170 - Return for new line
520  END
```

Initialization

Main Intergation Loop

All calculations of the distance from each charge and of the fields use an estimate of the position one-half step ahead of the present point on the line. This half-step method is quite accurate even with a relatively large step size (and hence relatively few iterations to complete a line). The question of accuracy and the idea of a half-step method are discussed further in Appendix B.

A RUN of this program looks like the following:

```
EMAP

(X,Y) STARTING PT. ON FIELD LINE?
?.55,.05
X             Y
 .879727       .425224
 1.13562       .853816
 1.2977        1.32582
 1.35226       1.82174
 1.28674       2.31608
 1.09079       2.77418
 .762383       3.14798
 .321695       3.37725
-.173504       3.40938
-.638683       3.23573
-1.00482       2.89915
-1.24142       2.46087
-1.34563       1.97332
-1.32623       1.47485
-1.1954        .993281
-.966379       .549795
-.655836       .158764
-.515865       .015919

(X,Y) STARTING PT. ON FIELD LINE?
?.55,.01
X             Y
 1.03884       .114918
 1.52333       .238369
 2.00277       .380176
 2.47694       .538725
 2.94573       .712554
```

```
 3.40907          .900448
 3.86689          1.1014
 4.31915          1.31457
 4.76582          1.53924
 5.20685          1.77478
 5.64222          2.02064
 6.07188          2.27634
 6.49579          2.54145
 6.91394          2.81558
 7.07958          2.92767

(X,Y) STARTING PT. ON FIELD LINE?
?
```

The field lines start at the positive charge and end at the negative charge. Some lines are much longer than others; the longer ones bend around from the back side of the positive charge to the back side of the negative charge. The line leaving the back side of the positive charge directly away from the negative charge must go through infinity to get back to the negative charge.

Practice Exercise: Modify the program to find field lines for four point charges: +1 at (+1, +1), -1 at (-1, +1), +1 at (-1, -1) and -1 at (+1, -1).

Plotting can be helpful in this exercise. If you have a plotter system or a plotting terminal, use it; if not, terminal plotting can serve. The changes required to use terminal plotting with EMAP refer to subroutines in Appendix A and are as follows:

```
GET-EMAP
10 READ X8,X9,Y8,Y9,Z0
20 DATA -5,5,-5,5,1
30 GOSUB 9000
205 GOSUB 9100
250
440 GOSUB 9100
490 GOSUB 9100
492 GOSUB 9200

APP-TTYPLO

RUN
EMAP

(X,Y) STARTING PT. ON FIELD LINE?
?.55,.05
MAX Y= 5
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Y
Y
Y
Y
Y
Y                        1 1  1
Y                      1        1
Y                               1
Y                     1           1
Y                    1
Y                                  1
Y                    1             1
Y                     1            1
Y                      1
Y                               1
Y                        1      1
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
MIN Y=-5  MIN X=-5             MAX X= 5
```

```
(X,Y) STARTING PT. ON FIELD LINE?
?.55,-.03

MAX Y= 5
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y                        1      1
Y                      1          1
Y                    1              1
Y                  1                  1
Y                                       1
Y                 1                       1
Y               1                           1
Y              1                              1
Y                                               1
Y             1                                 1
Y                                                1
Y             1                                  1
Y                                                 1
Y             1                                   1
Y                                                  1
Y              1                                   1
Y                                                  1
Y               1
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
MIN Y=-5  MIN X=-5             MAX X= 5


(X,Y) STARTING PT. ON FIELD LINE?
?.45,0
MAX Y= 5
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y                        1 1 1
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
MIN Y=-5  MIN X=-5             MAX X= 5

(X,Y) STARTING PT. ON FIELD LINE?
?
```

Sometimes it is useful to have several lines on one plot. The following changes in the original field line program, EMAP, listed on page 11, accomplish this.

```
GET-EMAP
10 READ X8,X9,Y8,Y9,Z0
20 DATA -5,5,-5,5,0
30 GOSUB 9000
182 IF X5=999 THEN 515
183 LET Z0=Z0+1
205 GOSUB 9100
250
440 GOSUB 9100
490 GOSUB 9100
515 GOSUB 9200

APP-TTYPLO
```

A RUN of this modified program is shown below:

```
RUN
EMAP

(X,Y) STARTING PT. ON FIELD LINE?
?.55,.05

(X,Y) STARTING PT. ON FIELD LINE?
?.55,-.03

(X,Y) STARTING PT. ON FIELD LINE?
?.45,0

(X,Y) STARTING PT. ON FIELD LINE?
?999,0
MAX Y= 5
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Y
Y
Y
Y
Y
Y                        1 1  1
Y                      1        1
Y                              1
Y                     1          1
Y                    1
Y                               1
Y                    1          1
Y                     1         1
Y                      1
Y                              1
Y                         23 3 32
Y                      2         2
Y                    2             2
Y                  2                 2
Y                                      2
Y                2                       2
Y              2                           2
Y             2                             2
Y                                            2
Y            2                               2
Y           2                                 2
Y                                              2
Y           2                                   2
Y           2                                   2
Y                                                2
Y                                               2
Y             2
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
MIN Y=-5  MIN X=-5            MAX X= 5
```

(The terminal plotting programming prints the last character plotted at each position.)

Finally, it is possible to modify the field line program to follow equipotential contours instead. All you do is move perpendicularly to the field $\vec{E}$ at each point (instead of parallel). The modified program is listed below, followed by a run and a terminal plot.

```
VMAP

100  LET N0=2
110  FOR J=1 TO N0
120  READ X[J],Y[J],Q[J]
130  NEXT J
140  DATA .5,0,1,-.5,0,-1
150  LET K0=9.E+09
160  LET D=.05
170  PRINT "(X,Y) STARTING PT. ON FIELD LINE?"
180  INPUT X5,Y5
190  LET X0=X5
200  LET Y0=Y5
210  LET X1=0
220  LET Y1=0
230  LET S0=.5
240  LET S=0
250  PRINT "X","Y"
260  LET E1=0
270  LET E2=0
280  FOR J=1 TO N0
290  LET R3=(X0+X1/2-X[J])*(X0+X1/2-X[J])
300  LET R3=R3+(Y0+Y1/2-Y[J])*(Y0+Y1/2-Y[J])
310  LET R3=R3↑1.5
320  LET E=K0*Q[J]/R3
330  LET E1=E1+(X0+X1/2-X[J])*E
340  LET E2=E2+(Y0+Y1/2-Y[J])*E
350  NEXT J
360  LET E0=SQR(E1*E1+E2*E2)
370  LET X1=-D*E2/E0
380  LET X0=X0+X1
390  LET Y1=D*E1/E0
400  LET Y0=Y0+Y1
410  LET S=S+D
420  IF S<S0 THEN 450
430  LET S=0
440  PRINT X0,Y0
450  IF ABS(X0-X5)+ABS(Y0-Y5)<.9*D THEN 470
460  IF ABS(X0)+ABS(Y0)<10 THEN 260
470  PRINT X0,Y0
480  PRINT
490  GOTO 170
500  END
```

```
VMAP

(X,Y) STARTING PT. ON FIELD LINE?
?.3,0
X             Y
 .655171      -.166054
 .54441        .219905
 .293041       3.07172E-02

(X,Y) STARTING PT. ON FIELD LINE?
?.1,0
X             Y
 .263204      -.459612
 .719458      -.610559
 1.12219      -.339661
 1.20455       .140735
 .921518       .536615
 .440887       .580765
 .127082       .214304
 9.85179E-02   1.68651E-02

(X,Y) STARTING PT. ON FIELD LINE?
?-.1,0
X             Y
-.263204      -.459612
-.719458      -.610559
-1.12219      -.339661
-1.20455       .140735
-.921519       .536616
-.440887       .580765
-.127082       .214304
-9.85179E-02   1.68656E-02

(X,Y) STARTING PT. ON FIELD LINE?
?
```

```
GET-VMAP
10 READ X8,X9,Y8,Y9,Z0
20 DATA -3,3,-3,3,0
30 GOSUB 9000
185 IF X5=999 THEN 495
186 LET Z0=Z0+1
205 GOSUB 9100
250
440 GOSUB 9100
470 GOSUB 9100
495 GOSUB 9200

APP-TTYPLO

RUN
VMAP

(X,Y) STARTING PT. ON FIELD LINE?
?.3,0

(X,Y) STARTING PT. ON FIELD LINE?
?.1,0

(X,Y) STARTING PT. ON FIELD LINE?
?-.1,0

(X,Y) STARTING PT. ON FIELD LINE?
?999,0
MAX Y= 3
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y                   3   3        2   2
Y
Y                 3          3 2    1     2
Y                            3 211
Y                                  1
Y                  3       3    2        2
Y                     3              2
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
MIN Y=-3  MIN X=-3             MAX X= 3
```

Practice Exercise: Annotate the program with comments by each line or group of lines explaining what calculation the line or group perform. You may use the annotations on the earlier program as a guide.

Practice Exercise: Modify either of the algorithmic programs so that the program follows either a field line or an equipotential. Allow the user to decide which type of contour to follow each time he starts a new contour.

CONCLUSION

The methods discussed here are very general. Field lines and equipotentials can be found for any source charge distribution. By writing and running your own programs for several different charge distributions, you will gain a very good feeling for the behavior of field lines and equipotentials. Field line mapping is fundamental to every part of electricity and magnetism, so it will be helpful to gain this intuition now.

CHAPTER ONE EXERCISES

1. Consider the line charge distribution: +3 at (+.5,0) and −1 at (−.5,0). (This potential is symmetric around the x-axis so we only need to view it in the x-y plane).

   a) *Find the potential for this charge distribution. Use the character plotting method to observe the potential. (You will need to print the values in the region first in order to know the largest and smallest values of the potential to plot.) Consider the regions:* $-.55 < x < -.45$, $-.05 < y < .05$; $+.45 < x < .55$, $-.05 < y < .05$; *and* $-5 < x < 5$, $-5 < y < 5$.

   b) *Sketch several equipotentials on each of the character plots. Since the change in potential is linear with the character set, if you sketch the equipotentials for every third character (eg., 2, 5, 8 etc.), then the contour lines are close together when the change in the potential is large (and the field is strong).*

   c) *Sketch four field lines on the character plots.*

   d) *Using an algorithmic method, compute four field lines for the large region considered in (a).*

   e) *Using an algorithmic method, compute four equipotential lines for the large region considered in (a).*

2. Consider the line charge distribution $\lambda = +1$ at (+.5, −.25), $\lambda = +1$ at (−.5, −.25) and $\lambda = +1$ at (0, +.62), an equilateral triangle. The line charges are perpendicular to the x-y plane.

   a) *Find the potential for this charge distribution. Use the character plotting method. Consider the region* $-2 < x < 2$, $-2 < y < 2$.

   b) *Sketch several equipotentials on the character plot.*

   c) *Sketch three field lines starting at each charge on the character plot.*

   d) *Using an algorithmic method, compute four field lines starting at each charge of this charge distribution.*

   e) *Using a algorithmic method, compute several equipotential lines. Pick equipotentials close to each charge and then far away from all of them.*

   f) *How would you compare the potential near each charge and far away from all of them to a single line charge distribution?*

3. Consider the line charge distribution of $\lambda = +1$ at (.5, .5); $\lambda = -1$ at (.5, -5); $\lambda = +1$ at (-.5, -.5); and $\lambda = -1$ at (-.5, +.5). The line charges are perpendicular to the x-y plane.

   a) *Find the potential for this charge distribution. Use the character plotting method. Consider the region $-2 < x < 2$, $-2 < y < 2$.*

   b) *Sketch several equipotentials on the character plot.*

   c) *Sketch three field lines starting at each charge on the character plot.*

   d) *Using an algorithmic method, compute four field lines for this charge distribution.*

   e) *Using an algorithmic method, compute several equipotential lines. Pick equipotentials close to each charge and then far away from all of them.*

   f) *Locate the planes on which the potential is zero.*

4. Modify Exercise 3 so that each of the line charges has $\lambda = +1$. Sketch your idea of the equipotential and the field lines before you calculate them.

   a) *Using an algorithmic method, compute four field lines starting at each positive charge for this charge distribution.*

   b) *Using an algorithmic method, compute equipotential lines for the distribution.*

   c) *How would you compare the potential near each charge and far away from all of them to a single charge distribution.*

5. Let us examine the idea that a set of point charges can be used to approximate any field distribution. To do this, let us approximate a line charge by a set of point charges. Consider the point charge distribution with nine +1 charges from (-2,0) to (2,0) at .5 intervals.

   a) *Compute the field line distribution for this charge distribution.*

   b) *Compute the equipotential lines for this distribution.*

   c) *Since you know the analytical expression for the field and potential from a line charge, compare your point charge approximation to it. Where does the point charge approximation to field lines and potential differ from the line charge? (There are three regions where the two differ most).*

   d) *How would you make a better approximation to the line charge?*

6. Consider the line charge distribution of $\lambda = +1$ at (0,0); $\lambda = -1$ at (-1,0); $\lambda = -1$ at (+1,0); $\lambda = +1$ at (-2,0); $\lambda = +1$ at (2,0); $\lambda = -1$ at (-3,0); $\lambda = -1$ at (3,0); $\lambda = +1$ at (-4,0); and $\lambda = +1$ at (4,0).

   a) *Compute the field line distribution for this charge distribution of $-.5 < x < .5$, $-1 < y < 1$.*

   b) *Compute the equipotential lines for this distribution.*

   c) *On what surfaces is the potential zero?*

7. In Exercises 3 and 6, and in the dipole example, the charge distribution leads to planes of zero potential. These problems illustrate the image phenomena. The field pattern of these charge distributions is the same as that of conducting sheets at zero potential and the charge distribution on one side. For example, Exercise 3 can be thought of as +1 at (.5, .5); -1 at (.5, -.5) and a conducting plane at zero potential along the y-axis. The way to solve the conducting sheet problem is to set it up as if each charge has an image of opposite sign an equal distance on the other side of the plane. Thus, Exercise 3 is the solution to the two charges and the conducting plane.

   a) *Exercise 3 is also the solution to another charge distribution and a plane. What is it?*

   b) *Exercise 3 can be seen as the solution to the charge distribution for two conducting planes and a point charge. Where are the planes?*

   c) *How can Exercise 6 be restated in terms of conducting planes?*

   d) *Consider the line charge distribution $\lambda = +2$ at (.5,0), $\lambda = -1$ at (.5, .5); $\lambda = -1$ at (.5, -.5) and a conducting plane along the yz axis. How would you solve this problem using the method of images?*

8. Consider the charge distribution caused by an infinite solid cylinder of charge centered on the z-axis with a radius of .5. This can be approximated by a set of line charges parallel to the z-axis.

   a) *Compute the field line distribution for this charge distribution.*

   b) *Compute the equipotential lines for this distribution.*

   c) *What is the potential inside the cylinder?*

   d) *What should the field be according to Gauss's law inside the cylinder? How does the potential and field outside of the cylinder compare to those of a single line charge?*

9. We have visualized fields by either equipotentials or field lines. We can also visualize fields by surfaces of constant field magnitude.

Consider the dipole charge distribution we used in the text.

a) Print the magnitude of the field over the region used in the text.

b) Convert the character printing program so that the characters represent magnitudes of fields.

c) Draw in the equifield regions on the character plots.

d) How do the equifield regions differ from equipotential lines?

e) Is there any relationship between the constant field contours and the directions in which the fields point?

10. The analytical solution of a dipole is:

the potential $V = p \cos\theta / r^2$

and $E_x = 3p \sin\theta \cos\theta / r^3$

$E_z = p\,(3\cos^2\theta - 1)/r^3$

are the field components when the dipole p is centered at the origin and is parallel to the z direction. The angle θ is measured with respect to the z direction, and r is measured from the center of the dipole.

If the dipole is caused by two single charges on the z-axis, then $|p| = qs$ where s is the distance between the charges (the positive charge is assumed to be in the positive z direction).

a) Compare the program results obtained for the potential calculated from two point charges with the analytical dipole approximation. (Compare the values on the x- and z-axes at the distances 1, 5, 20.)

b) Compare the calculated results for the field with the analytical approximation. To do this, see if the two solutions are parallel to each other along the x-axis at 1, 5, 20 and along the 45° line between the x- and z-axes at the distances 1, 5, 20 from the origin.

CHAPTER TWO: MAGNETOSTATIC FIELDS

INTRODUCTION

Magnetic fields $\vec{B}(\vec{r})$ result from currents. (We will use the term magnetic field even though magnetic induction is strictly correct; the distinction is really only important when you deal with magnetic materials.) If the currents are steady (that is, do not change with time), the magnetic field is magnetostatic. The purpose of this chapter is to find how to map magnetostatic fields by means of flux lines.

MAGNETIC FIELDS FOR VARIOUS CURRENT DISTRIBUTIONS

Physics textbooks derive equations for the magnetic field due to simple current distribution, such as the long straight wire. The texts also introduce the Biot-Savart law (sometimes called the Biot law) which gives the magnetic field everywhere in space due to a short segment of current, $Id\vec{\ell}$. These are the only results for magnetic field that we will need in this chapter. The Biot-Savart law is sufficiently general and, when you use a computer, sufficiently easy to use that any current distribution can be handled.

For a long straight wire the magnetic field is given by $\vec{B} = 2k_0\ I/r$ (tangential around the wire with sense by the right-hand rule), where r is the perpendicular distance to the wire, and k_0 is a constant which determines the units. $k_0 = \mu_0/4\pi = 10^{-7}$ in mks units in which B is measured in Tesla; $k_0 = 1/c = 1/3\text{x}10^{10}$ in Gaussian (cgs) units in which B is measured in Gauss. In mixed units where currents are measured in amperes and magnetic field in Gauss, $k_0 = .1$. (One Tesla is 10^4 Gauss.)

The Biot-Savart law states that the magnetic field, $d\vec{B}$, due to a small element of current, $Id\vec{\ell}$, is

$$d\vec{B} = k_0\ I\ d\vec{\ell} \times \vec{r}/|\vec{r}|^3 \qquad (7)$$

where $\vec{r}$ is the distance from $d\vec{\ell}$ to the point, (x, y, z), that the field, $d\vec{B}$, is measured. This relationship allows you to compute the magnetic field due to any circuit by breaking the circuit up into little pieces and then adding up all the $d\vec{B}$'s. This relationship for $d\vec{B}$ is an inverse square law but has the complication of the vector cross product in the numerator. The cross product (in Cartesian coordinates especially) is quite easy to calculate on a computer.

MAPPING FIELD LINES

There are two ways to visualize magnetic field patterns. The first is not often used but is easy to understand. In this first method, the magnetic field, $\vec{B}$, is computed at a large number of points throughout the region of interest. The entire field is then visualized in terms of contours (or surfaces in three dimensions) of constant $|\vec{B}|$. This method is not in common use but is considered further in a problem at the end of the chapter.

The second (and very common) way to visualize magnetic field patterns uses flux lines. The definition of a flux line has two parts: (1) the magnetic field, $\vec{B}(\vec{r})$, is tangential to the flux line at each point, $\vec{r}$, and (2) the number of flux lines passing through a (unit, perpendicular) cross section placed at any point, $\vec{r}$, is proportional to the magnitude of the magnetic field, $|\vec{B}(\vec{r})|$, at that point.

To calculate the shapes of flux lines analytically (the result then being an equation for the lines) is very difficult. With a computer, such calculations are straightforward. Suppose you know a point, (x, y, z), on a flux line. (Any point (x, y, z) lies on some flux line. So choosing an arbitrary starting point defines a particular flux line which will be followed.) Suppose further that you want to take a step of size Δs along the flux line. The first defining property for flux lines ($\vec{B}$ is parallel to the line) implies that

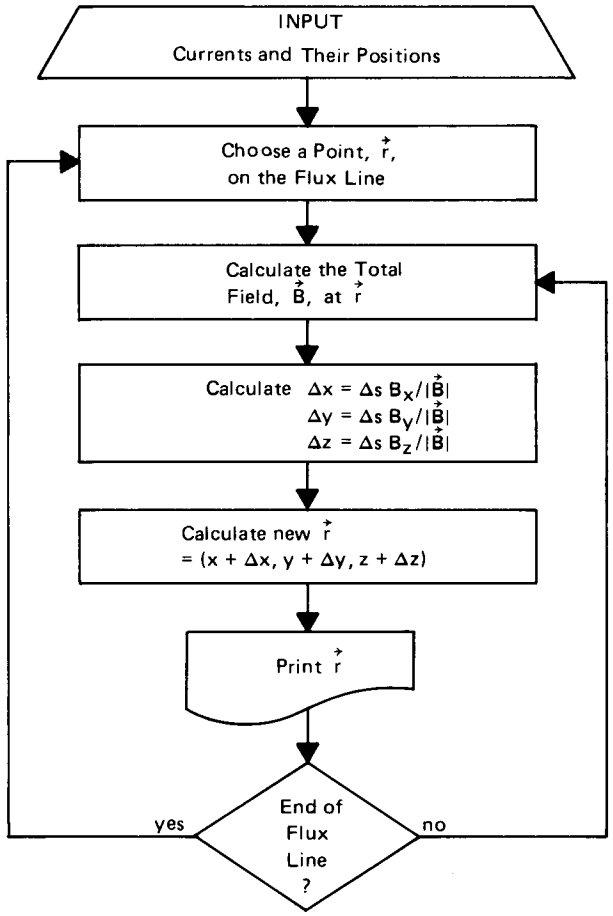
$$\Delta x = \Delta s \, B_x/|\vec{B}|$$

$$\Delta y = \Delta s \, B_y/|\vec{B}|$$

$$\Delta z = \Delta s \, B_z/|\vec{B}|$$

These equations say that $\vec{\Delta s} = \vec{\Delta x} + \vec{\Delta y} + \vec{\Delta z}$ is parallel to $\vec{B} = \vec{B}_x + \vec{B}_y + \vec{B}_z$. At the point (x, y, z) you calculate $\vec{B}$; and then calculate Δx, Δy and Δz. The next point along the flux line is $(x + \Delta x, y + \Delta y, z + \Delta z)$. Since you now know another point on the flux line, you can repeat the process. In this way the computer is used to walk along the flux line in small steps, Δs.

Flux lines never end, so the process continues until we return to the starting point. A block diagram of this strategy looks like this:



The fact that flux lines never end is a consequence of the fact that the next flux through any region must be zero ($\oint \vec{B} \cdot d\vec{A} = 0$). As much flux must enter as leaves; hence, no flux line can end. (The corresponding statement for electrostatic fields is $\oint \vec{E} \cdot d\vec{A} = 4\pi k_0 q$. This implies that electrostatic field lines end or start only on charges, which may, if necessary, be at infinity.)

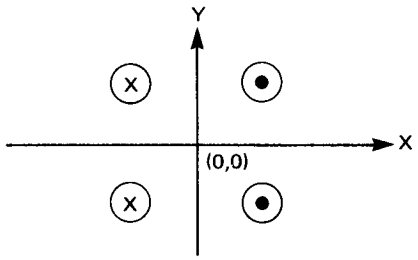
Notice that this iterative procedure for following flux lines will break down if $|\vec{B}| = 0$ at any point on the line.

FLUX LINES FOR A SET OF LONG STRAIGHT WIRES

Let's consider a simple case.

Example 1. Map the magnetic induction pattern due to four long straight wires parallel to the z-axis. The currents and the points (x, y) at which the wires cut the x-y plane are: +1 at (+1, +1); -1 at (-1, +1); -1 at (-1, +1); and +1 at (+1, -10).

Schematically the situation in the x-y plane is:



This is a relatively simple current distribution, but analytical methods cannot calculate the flux lines. A plus one current means a current of one ampere (in mks units) which is coming out of the page (by the right-hand rule).

By symmetry, the magnetic field, $\vec{B}$, has no z-component. All planes parallel to the x-y plane are equivalent, so we need only consider the x-y plane itself. A program implementing the strategy discussed above might look like the following:

```
EMAP

100  DIM X[4],Y[4],I[4]                          ┐
110  LET N0=4                                    │
120  FOR J=1 TO N0                               │ Currents and
130  READ X[J],Y[J],I[J]                         │ Geometry for
140  NEXT J                                      │ long, straight
150  DATA 1,1,1,-1,1,-1,-1,-1,1,-1,1,1           │ wires
160  LET K0=1.E-07 - Units                       │
170  LET D=.05 - Step size, Δs                   ┘
180  PRINT "(X,Y) STARTING PT. ON FIELD LINE?"
190  INPUT X5,Y5
200  LET X0=X5 - x
210  LET Y0=Y5 - y
220  LET X1=0 - Δx
230  LET Y1=0 - Δy
240  LET S0=.5 - Print every S0 along line
250  LET S=0 - Distance, S, along line
260  PRINT "X","Y"
270  LET B1=0                                    ┐
280  LET B2=0                                    │
290  FOR J=1 TO N0                               │
300  LET R2=(X0+X1/2-X[J])*(X0+X1/2-X[J])        │ |r|²
310  LET R2=R2+(Y0+Y1/2-Y[J])*(Y0+Y1/2-Y[J])     ┘
320  LET B=2*K0*I[J]/R2 - |B|/r                  │ Calculate B
330  LET B1=B1-(Y0+Y1/2-Y[J])*B - Bx             │ (half-stepped)
340  LET B2=B2+(X0+X1/2-X[J])*B - By             │
350  NEXT J                                      ┘
360  LET B0=SQR(B1*B1+B2*B2) - |B|
370  LET X1=D*B1/B0 - Δx
380  LET X0=X0+X1 - New x
390  LET Y1=D*B2/B0 - Δy
400  LET Y0=Y0+Y1 - New y
410  LET S=S+D                                   ┐
420  IF S<S0 THEN 450                            │ Print out
430  LET S=0                                     │ Lines
440  PRINT X0,Y0                                 ┘
450  IF ABS(X0)+ABS(Y0)>10 THEN 470 - Far off page   ┐ Tests for
460  IF ABS(X0-X5)+ABS(Y0-Y5)>.9*D THEN 270 - Back to start ┘ End of Line
470  PRINT X0,Y0 - Last point on line
480  PRINT
490  GOTO 180 - Return for new line
500  END
```

The calculation of the magnetic field, $\vec{B}(\vec{r})$, in this program uses a point (approximately) one-half a step ahead of the present point on the line. A discussion of convergence and the half-step method are found in Appendix B. A run of this program looks like this:

```
RUN
BMAP

(X,Y) STARTING PT. ON FIELD LINE?
?.8,.8
X              Y
 1.2319         .640183
 1.40682        1.06023
 1.0171         1.30555
 .743917        .945335
 .792453        .804443

(X,Y) STARTING PT. ON FIELD LINE?
?.5,0
X              Y
 .435075       -.494747
 .382484       -.990777
 .496948       -1.4741
 .790006       -1.87544
 1.20665       -2.14756
 1.68808       -2.27552
 2.18638       -2.26245
 2.66452       -2.12079
 3.09446       -1.8678
 3.45537       -1.52326
 3.7323        -1.10812
 3.9151        -.643715
 3.99767       -.151479
 3.97737        .347228
 3.85484        .831059
 3.63404        1.27863
 3.3226         1.66852
 2.93244        1.97947
 2.48067        2.19089
 1.99078        2.28379
 1.49415        2.24244
 1.03185        2.05738
 .656388        1.73115
 .428638        1.28974
 .388341        .794198
 .471962        .301455
 .500743        3.20055E-03

(X,Y) STARTING PT. ON FIELD LINE?
?
```

Practice Exercise: Modify this program to follow flux lines due to two wires both with I = +1 with (x,y) position (-.5,0) and (+.5,0).

Terminal plotting is sometimes useful to display flux lines. If you have a plotter system or a plotting terminal, you may want to use that. Terminal plotting can be performed as discussed in Appendix A. The necessary changes in the flux line program BMAP on page 24 (and then a run) to do terminal plotting might look like:

```
GET-BMAP

10 READ X8,X9,Y8,Y9,Z0
20 DATA -5,5,-5,5,1
30 GOSUB 9000
205 GOSUB 9100
250
430 GOSUB 9100
450 GOSUB 9100
452 GOSUB 9200
```

```
RUN
BMAP

(X,Y) STARTING PT. ON FIELD LINE?
?.7,.7
MAX Y= 5
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y                              1  1 1
Y                             1
Y                             1     1
Y                               1   1
Y                              1 1
Y                              1     1
Y                                    1
Y                             1
Y                              1 1 1
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
MIN Y=-5  MIN X=-5            MAX X= 5

(X,Y) STARTING PT. ON FIELD LINE?
?.5,0
MAX Y= 5
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Y
Y
Y
Y
Y
Y
Y
Y
Y                                 1  1 1
Y                               1          1
Y                             1              1
Y                            1                1
Y
Y                            11                 1
Y                            1                   1
Y                             1                  1
Y                            1
Y                                               1
Y                            1                  1
Y                            1
Y                                             1
Y                              1 1       1 1
Y                                  1   1
Y
Y
Y
Y
Y
Y
Y
Y
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
MIN Y=-5  MIN X=-5            MAX X= 5

(X,Y) STARTING PT. ON FIELD LINE?
?
```

Practice Exercise: By hand, plot the positions of the line currents on the terminal plots.

The flux lines bend around and close on themselves. The field is higher halfway between the oppositely-directed currents and lower halfway between the like-directed currents. The field is quite uniform (that is, the flux lines are nearly parallel over a fair-sized region) at the center of the pattern.

THE DENSITY OF FLUX LINES

If the flux lines are started correctly somewhere in the pattern, the density of the lines will be correct everywhere. That is, if the second defining property is obeyed somewhere by all the flux lines, it will be obeyed everywhere. The difficulty is that there is no easy way to make the density correct anywhere. (This is unlike the case for electrostatic field lines in Chapter One. For electrostatic field lines you can start the lines correctly near the charges.)

You might be interested in thinking about this difficulty. Can you see a way to start the lines with the correct density? How about calculating the flux through small regions across, say, the line from (-1,0) to (+1,0) in our example? You might then start the correct number of lines in each region.

FIELD LINES FOR COMPLICATED CURRENT DISTRIBUTIONS

The same algorithmic procedure to follow flux lines can be used for more complicated current distributions. You can use the Biot-Savart law quoted at the beginning of this chapter. By breaking up any current distribution into little pieces, $I\vec{d\ell}$, using the Biot-Savart law to calculate the $\vec{dB}$s for each piece, and then adding the $\vec{dB}$s to find the total magnetic field, $\vec{B}$, you can calculate the magnetic field anywhere in space for any current distribution. Using the resulting values of $\vec{B} = (B_x, B_y, B_z)$, you can follow the flux lines for any current distribution by making Δs parallel to $\vec{B}$ as above.

Let's find the field pattern for a Helmholtz pair of circular coils. A Helmholtz pair has two parallel, circular coils carrying equal currents and having a separation equal to the coils' (common) radius.

Example 2. Map the magnetic field pattern in the x-y plane due to a pair of Helmholtz coils of radius one carrying a current of +1A. Both coils are parallel to the y-z plane.

A program which calculates the magnetic field, $\vec{B}$, using the Biot-Savart law and follows flux lines algorithmically might look like the following:

```
COILMP

100  DIM C[20],S[20]
110  LET N0=2
120  FOR J=1 TO N0
130  READ X[J],R[J],I[J]
140  NEXT J
150  DATA .5,1,1,-.5,1,1
160  LET P2=6.28318
170  LET N9=10
180  FOR I=1 TO N9                    ┐
190  LET T=P2*(I-.5)/N9               │ Store
200  LET C[I]=COS(T)                  │ sines and
210  LET S[I]=SIN(T)                  │ cosines
220  NEXT I                           ┘
230  LET K0=1.E-07
240  LET D=.1
250  PRINT "(X,Y) STARTING PT. ON FIELD LINE?"
260  INPUT X5,Y5
270  LET X0=X5
280  LET Y0=Y5
290  LET X1=0
300  LET Y1=0
310  LET S0=.5
320  LET S=0
330  PRINT "X","Y"
340  LET B1=0
350  LET B2=0
360  LET B3=0
370  FOR J=1 TO N0 – step through coils
380  FOR I=1 TO N9 – step around each coil
390  LET L1=0
400  LET L2=-P2*R[J]*S[I]/N9
410  LET L3=P2*R[J]*C[I]/N9
420  LET L7=X[J]
430  LET L8=R[J]*C[I]
440  LET L9=R[J]*S[I]
450  LET X6=X0+X1/2-L7
460  LET Y6=Y0+Y1/2-L8
470  LET Z6=-L9
480  LET R6=SQR(X6*X6+Y6*Y6+Z6*Z6)
490  LET R6=R6*R6*R6
500  LET C1=L2*Z6-L3*Y6               ┐
510  LET C2=L3*X6-L1*Z6               │ Cross product
520  LET C3=L1*Y6-L2*X6               ┘
530  LET B1=B1+K0*I[J]*C1/R6 – Bx
540  LET B2=B2+K0*I[J]*C2/R6 – By
550  LET B3=B3+K0*I[J]*C3/R6 – Bz
560  NEXT I
570  NEXT J
580  LET B0=SQR(B1*B1+B2*B2+B3*E3)
590  LET X1=D*B1/B0
600  LET X0=X0+X1
610  LET Y1=D*B2/B0
620  LET Y0=Y0+Y1
630  LET S=S+D
640  IF S<S0 THEN 670
650  LET S=0
660  PRINT X0,Y0
670  IF ABS(X0)+ABS(Y0)>10 THEN 690
680  IF ABS(X0-X5)+ABS(Y0-Y5)>.9*D THEN 340
690  PRINT X0,Y0
700  PRINT
710  GOTO 250
720  END
```

Practice Exercise: Annotate the program by written comments by each line.

A run looks like this:

```
RUN
COILMP

(X,Y) STARTING PT. ON FIELD LINE?
?0,1
X                Y
 .469244          .857036
 .789097          1.18179
 .615597          1.63534
 .180641          1.86649
-.312249          1.83018
-.700194          1.52856
-.757882          1.05049
-.341947          .869398
-6.54586E-02      .985284

(X,Y) STARTING PT. ON FIELD LINE?
?.4,.9
X                Y
 .760749          1.13971
 .596547          1.59283
 .152504          1.80504
-.33987           1.75606
-.711667          1.43789
-.684021          .965799
-.226359          .939789
 .235256          .932256
 .325676          .889544

(X,Y) STARTING PT. ON FIELD LINE?
?0,.5
X                Y
 .499765          .4973
 .9769            .635058
 1.37464          .935003
 1.68635          1.32485
 1.92196          1.76524
 2.08598          2.23709
 2.17837          2.72804
 2.1968           3.22724
 2.13798          3.72326
 1.99862          4.20284
 1.77646          4.65
 1.47184          5.04543
 1.09004          5.3667
 .644055          5.5901
 .156582          5.69515
-.341449          5.6708
-.816714          5.5197
-1.24089          5.25712
-1.59496          4.90542
-1.86911          4.48822
-2.06025          4.0269
-2.16923          3.53949
-2.19883          3.04085
-2.15247          2.54345
-2.03318          2.05834
-1.84265          1.59658
-1.57955          1.17213
-1.23706          .809351
-.806264          .56055
-.313676          .492408
-1.38063E-02      .50097

(X,Y) STARTING PT. ON FIELD LINE?
?
```

And, finally, a terminal plot of the program, COILMP on page 28, is shown below:

```
GET-COILMP

10 READ X8,X9,Y8,Y9,Z0
20 DATA -5,5,-5,5,0
30 GOSUB 9000
265 IF X5=999 THEN 715
266 LET Z0=Z0+1
285 GOSUB 9100
330
660 GOSUB 9100
690 GOSUB 9100
715 GOSUB 9200
APP-TTYPLO

RUN
COILMP

(X,Y) STARTING PT. ON FIELD LINE?
?0,1

(X,Y) STARTING PT. ON FIELD LINE?
?.4,.9

(X,Y) STARTING PT. ON FIELD LINE?
?0,.5

(X,Y) STARTING PT. ON FIELD LINE?
?999,0
MAX Y= 5
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Y                    3
Y                                  3
Y                   3                3
Y                  3
Y                3                    3
Y                                     3
Y                3
Y                3                    3
Y                                     3
Y                 3          1
Y                  3    1 2  2 2      3
Y                    3   2       1   3
Y                        1212122 2  3
Y                     3 3   3    3
Y                        3   3
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
Y
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
MIN Y=-5  MIN X=-5            MAX X= 5
```

Again, the strategy used in this program is entirely general. The magnetic field, $\vec{B}(\vec{r})$, for any current distribution can be calculated from the Biot-Savart Law: sets of short wire segments; open-wound, short solenoids; sets of square coils; anything at all. A number of possible cases are treated in the exercises at the end of the chapter.

CONCLUSION

This chapter has introduced a general method to map magnetic field patterns. Both simple sets of long straight wires were discussed and a completely general procedure based on the Biot-Savart law was presented. By running several situations you can get a good intuitive feel for the behavior of magnetic fields.

CHAPTER TWO EXERCISES

1. Consider two long, straight, current-carrying wires parallel to the z-axis. The currents and the points (x,y) at which the wires cut the x-y plane are: +1 at (1,0); +1 at (-1,0).

   a) *Determine the magnetic field along the x-axis from -3 to 3.*

   b) *In the region $.5 < x < 1.5$, $-.5 < y < 5$, calculate six magnetic flux lines such that the number of flux lines through a given area on the x-axis is proportional to the flux in that region. (Since the field is independent of z, the number of flux lines through a Δx should be proportional to the average magnetic field on that region.)*

   c) *In the region $-3 < x < 3$, $-3 < y < 3$, calculate six magnetic flux lines such that the number of flux lines through a given region is proportional to the field in that region.*

   d) *In the region close to each wire, how does the field compare to that of a single current-carrying wire? In the region far away from all the wires, how does the field compare to that of a single current-carrying wire?*

2. Consider two long current-carrying wires parallel to the z-axis. The currents and the points (x,y) at which the wires cut the x-y plane are: +1 at (1,0); - 1 at (- 1,0).

   a) *Determine the magnetic field along the x-axis from - 3 to 3.*

   b) *In the region $.5 < x < 1.5$, $-.5 < y < 5$, calculate six magnetic flux lines such that the number of flux lines through a given area on the x-axis is proportional to the flux in that region. (Since the field is constant in the z direction the number of flux lines through a Δx should be proportional to the average magnetic field on that region.)*

   c) *In the region $-3 < x < 3$, $-3 < y < 3$, calculate six magnetic flux lines such that the number of flux lines through a given region is proportional to the field in that region.*

   d) *In the region close to each wire, how does the field compare to that of a single current-carrying wire? In the region far away from all the wires, how does the field compare to a single current-carrying wire?*

3. Consider the simple case of a single loop of current-carrying wire. Say the loop lies in the y-z plane centered at the origin with a radius of one and carries a current of one.

   a) *Using the Biot-Savart law and the computer integration method, calculate the magnetic field at several points along the x-axis. (x = 0, 1, 5, 10).*

b) *It is possible to calculate these axial fields analytically. Do so and compare the analytical results with the ones obtained from the computer method.*

c) *Evaluate the magnetic field along the y-axis out to the edge of the coil.*

d) *Calculate the flux line pattern in the y-z plane through the coil.*

4. Consider the case of four current-carrying wires parallel to the z-axis. The currents and the points (x,y) at which the wires cut the x-y plane are: +1 at (1.5,1); -1 at (-1.5,1); +1 at (.5,-1); and -1 at (-.5,-1).

a) *Sketch the flux lines close to each wire. Now sketch the whole flux line distribution.*

b) *Calculate the flux line pattern for this distribution.*

5. Consider the case of four current-carrying wires parallel to the z-axis. The currents and the points (x,y) at which the wires cut the x-y plane are +3 at (2,0); -3 at (-2,0); -1 at (1,0) and +1 at (-1,0).

a) *Sketch the flux lines close to each wire. Now sketch the whole flux line distribution.*

b) *Calculate the flux line pattern for this distribution.*

6. Consider a square Helmholtz pair of coils. Let the corners of the square loops be defined by the (x,y,z) points: (+.5,1,1), (.5,1,-1), (.5,-1,-1), (.5,-1,1) and (-.5,1,1), (-.5,1,-1), (-.5,-1,-1), (-.5,-1,1).

a) *Calculate the flux line pattern for this distribution. Since it is not rotationally symmetric about the x-axis, calculate it in the x-z plane and then in the plane defined by the point (0,1,1) and the z-axis (i.e., the plane at 45° to both the x-z and y-z planes).*

b) *Compare the axial field to that of the true (circular) Helmholtz pair. Compare both the magnitude of the field and the uniformity of the field in the center of each pair.*

7. Consider an infinite current-carrying cylinder of radius one. Model the cylinder as sixteen infinitely long wires parallel to the z-axis.

a) *Calculate the field distribution for this pattern.*

b) *How well does this compare to the Biot-Savart law inside the cylinder? Outside the cylinder how does it compare in magnitude and shape to a single wire at the origin with a current of sixteen?*

8. The field for an electric dipole was given in Chapter 1, Exercise 10. How well does the magnetic field of a simple loop in the x-y plane of radius 1 compare to the dipole equation when you are far away from the loop? Use the distance of ten for your comparison. What dipole moment, p, is necessary to make the magnitudes agree? How does this dipole moment vary with the current in the loop?

9. Flux lines are not the only method of studying magnetic fields. Write a program similar to the equipotential character mapping program to print like characters for regions with the magnitude of $|\vec{B}|$ equal. This method could be useful for determining regions of constant field.

   a) Use your program to study the field pattern in Exercise 1.

   b) Use your program to study the field pattern in Exercise 2.

10. Consider a solenoid of radius 1 along the x-axis five units long centered at the origin. (Approximate the solenoid by five current-carrying loops).

    a) Calculate the field line distribution.

    b) Using the field line mapping program or the field magnitude program of Exercise 9, determine what current should be run through the end coils to make the field inside the solenoid more uniform.

CHAPTER THREE: THE MOTION OF CHARGED PARTICLES IN ELECTRO-STATIC AND MAGNETOSTATIC FIELDS

INTRODUCTION

Many students believe that charged particles move along field lines. Except in very rare instances, charged particles do not move along field lines. Although the direction of the field line at any point defines the direction of the force at that point, the force only changes the velocity not the position itself. Once it is moving, the charged particle will, in general, cross field lines because of inertia. (If the particle moves through a very viscous medium, you can force the particle to stay on a field line.)

This chapter introduces a way to calculate the trajectories of charged particles in any (combination of) electrostatic and magnetostatic fields. Those of you who have seen $\vec{F} = m\vec{a}$ algorithmic solutions with a computer before (probably in mechanics) will recognize the discussion in this chapter. The chapter is based on an algorithmic solution for particle trajectories when the forces are due to $\vec{E}$ and $\vec{B}$.

The forces on charged particles in electrostatic and magnetostatic fields are discussed in physics textbooks. The force on a charged particle in an electrostatic field, $\vec{E}$, is just $\vec{F}(\vec{r}) = q\vec{E}(\vec{r})$, where q is the value of the charge. The force on a charged particle in a magnetic field (or, more correctly, magnetic induction), $\vec{B}$, is the Lorentz force, $\vec{F}(\vec{r}) = q(\vec{v}(\vec{r}) \times \vec{B}(\vec{r}))$. In combined electrostatic and magnetostatic fields, the force is $\vec{F} = q(\vec{E}+\vec{v}\times\vec{B})$.

AN ALGORITHMIC SOLUTION TO NEWTON'S SECOND LAW

$\vec{F} = m\vec{a}$ is Newton's Second Law and is the basis of classical calculations of particle trajectories. In most physical situations, you know the force, $\vec{F}(\vec{r})$. Hence, you know the acceleration $\vec{a}(\vec{r}) = \vec{F}(\vec{r})/m$. Given a way to find the acceleration at any point in space, the computer can easily be programmed to find the trajectory of any particle.

Suppose you know a point, $\vec{r} = (x,y,z)$, on the trajectory of a particle and also know the particle's velocity, $\vec{v} = (v_x, v_y, v_z)$ at that point. (You usually know $\vec{r}$ and $\vec{v}$ at the initial point of the trajectory.) Suppose you want to find the position a short time, Δt, later. From the acceleration, you can find the change in the velocity of the particle in the time, Δt: $\Delta\vec{v} = \vec{a}\Delta t$. Then, the new velocity after the time, Δt, is

$$\vec{v}_{new} = \vec{v}_{old} + \Delta\vec{v} \qquad (8)$$

From the velocity, you can find the change in position $\vec{r}$ during the time Δt, since

$$\Delta\vec{r} = \vec{v}\Delta t \qquad (9)$$

Hence, the next position on the particle's trajectory is

$$\vec{r}_{new} = \vec{r}_{old} + \vec{v}\Delta t \qquad (10)$$

The new time is $t_{new} = t_{old} + \Delta t$.

Now that you know another point on the trajectory, the process can be repeated over and over. The result is the trajectory of the particle as it moves through the force, $\vec{F}(\vec{r})$. We will discuss two cases: the first is the motion of charged particles in electrostatic fields, and the second is the motion of charged particles in combinations of electrostatic and magnetostatic fields. The second case is somewhat more complicated because the force on a charged particle in a magnetic field is velocity-dependent.

THE MOTION OF A CHARGED PARTICLE IN AN ELECTROSTATIC FIELD, $\vec{E}$

The force on a charged particle in an electrostatic field, $\vec{E}$, is just $\vec{F} = q\vec{E}$. You can calculate the electrostatic field, $\vec{E}$, for any distribution of point source charges and line source charges. The field for a point source charge is $\vec{E} = k_0\, Q/r^2\ (\vec{r}/r)$ where Q is the value of the source charge and $\vec{r}$ is the distance from the source charge, $r = |\vec{r}|$. The field for a line source charge is $\vec{E} = k_0\lambda/r\ (\vec{r}/r)$ where λ is the charge per unit length of the line charge and r is the perpendicular distance from the line charge. k_0 defines the units; $k_0 = 9 \times 10^9$ in mks units and $k_0 = 1$ for cgs units. Any general source charge distribution can be treated as a set of point or line charges.

Example 1. Find the trajectory of a positron (a particle with the mass of an electron but opposite (positive) charge) moving under the influence of a uniform $\vec{E}$ field parallel to the y-axis. Start the trajectory at the origin with velocity (4,0,0).

One program which implements the algorithmic solution discussed above is shown below:

```
MTONE

100  READ E1,E2,E3 ] Set E field
110  DATA 0,1,0    ]
120  PRINT "INITIAL X,Y,Z,VX,VY,VZ & FINAL T?" ] Initial r, v and
130  INPUT X,Y,Z,V1,V2,V3,T0                    ] final time
140  LET D=.05 - Δt
150  LET Q0=1 - q/m
160  LET T=0
170  LET T9=0
180  REM COMPUTE FIELDS, ACCELS., VEL., POSITION, TIME
190  LET A1=Q0*E1 - a_x
200  LET A2=Q0*E2 - a_y
210  LET A3=Q0*E3 - a_z
220  IF T>0 THEN 270 ]
230  LET V1=V1+A1*D/2 |
240  LET V2=V2+A2*D/2 | Initial half step
250  LET V3=V3+A3*D/2 |
260  GOTO 300        ]
270  LET V1=V1+A1*D - V_x
280  LET V2=V2+A2*D - V_y
290  LET V3=V3+A3*D - V_z
300  LET X=X+V1*D - New X
310  LET Y=Y+V2*D - New Y
320  LET Z=Z+V3*D - New Z
330  LET T=T+D - New t
340  LET T9=T9+D                                  ]
350  IF ABS(T9-T0/10)/T9>.000001 THEN 380         | Print results
360  LET T9=0                                     | every T0/10
370  PRINT T;X;Y;Z                                ]
380  IF T<T0 THEN 180 - Return for next point on trajectory
390  PRINT
400  GOTO 120 - Return for new trajectory
410  END
```

The program uses a half-step method (the velocity is one-half a time step, Δt, away from the position). This half-step method increases the convergence of the procedure and is discussed further in Appendix B. The units used in the program are normalized (because q/m is set to 1) so that fields are measured in units of the charge-to-mass ratio of the particle. For example, since the particle is a positron, q/m = $+1.759 \times 10^{11}$ in mks units, so a field with $|\vec{E}| = 1$ actually has $|\vec{E}| = 5.685 \times 10^{-12}$ V/m.

A run of this program looks like the following:

```
RUN
MTONE

INITIAL X,Y,Z,VX,VY,VZ & FINAL T?
?0,0,0,4,0,0,10
 1. 4. .5 0
 2. 7.99999 2. 0
 3. 12. 4.5 0
 4.00001 16. 8. 0
 5.00001 20. 12.5 0
 6.00001 24.0001 18. 0
 7.00002 28.0001 24.5 0
 8.00002 32.0001 32.0001 0
 9.00001 36. 40.5001 0
 9.99999 40. 50.0001 0

INITIAL X,Y,Z,VX,VY,VZ & FINAL T?
?
```

You can plot the trajectory of the particle using MTONE and the terminal plotting discussed in Appendix A. The required changes in the program and a run of the modified program are shown below:

```
GET-MTONE

10 READ X8,X9,Y8,Y9,Z0
20 DATA 0,50,0,50,1
30 GOSUB 9000
132 LET X0=X
134 LET Y0=Y
136 GOSUB 9100
370 LET X0=X
372 LET Y0=Y
374 GOSUB 9100
390 GOSUB 9200

APP-TTYPLO

RUN
MTONE
INITIAL X,Y,Z,VX,VY,VZ & FINAL T?
?0,0,0,4,0,0,10
MAX Y= 50
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Y
Y
Y
Y
Y
Y
Y                                        1
Y
Y
Y
Y
Y                                     1
Y
Y
Y
Y                                  1
Y
Y
Y
Y                               1
Y
Y
Y                            1
Y
Y
Y                         1
Y
Y                      1
Y
Y                   1
Y
Y              1
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
```

```
MIN Y= 0  MIN X= 0            MAX X= 50
INITIAL X,Y,Z,VX,VY,VZ & FINAL T?
?
```

For a uniform electrostatic field, the trajectory is a parabola (as can be derived analytically). Consequently, the motion is just like that in a uniform gravitational field. Thus, this particular example can be used to check the numerical method. This check is pursued in a problem at the end of the chapter.

Practice Exercise: Modify the program, MTONE, so that it calculates the electrostatic field inside the integration loop. Use the electrostatic field due to a point charge at the origin.

A terminal plot of several orbits in the field of a −1 charge at the origin looks like:

```
10  READ X8,X9,Y8,Y9,Z0
20  DATA -2,3,-2.5,2.5
30  GOSUB 9000
100
110
131  IF X=999 THEN 405
132  LET Z0=Z0+1
133  LET X0=X
134  LET Y0=Y
185  LET R3=((X1+X1/2)↑2+(Y1+Y1/2)↑2)↑1.5
190  LET A1=0
200  LET A2=0
210  LET A3=0
370  LET X0=X
372  LET Y0=Y
374  GOSUB 9100
390
405  GOSUB 9200

APP-TTYPLO

RUN
MTONE

INITIAL X,Y,Z,VX,VY,VZ & FINAL T?
?-1,0,0,0,1,0,7
INITIAL X,Y,Z,VX,VY,VZ & FINAL T?
?-1,0,0,0,1.2,0,15
INITIAL X,Y,Z,VX,VY,VZ & FINAL T?
?999,0,0,0,0,0,0
MAX Y= 2.5
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Y
Y
Y
Y
Y
Y                                2
Y
Y                  2                       2
Y
Y                  1
Y                           1
Y            1     •                            2
Y
Y                                  1
Y
Y          2                                      2
Y
Y                                  1
Y
Y            1                                    2
Y                           1
Y                  1
Y
Y                  2                       2
Y
Y                                  2
Y
Y
Y
Y
Y
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
MIN Y=-2.5  MIN X=-2          MAX X= 3

END
```

Practice Exercise: Plot the position of the source charge on the terminal plot.

These trajectories can also be calculated analytically, although with some difficulty. The trajectories are exactly similar to the orbits of a satellite around the earth.

Practice Exercise: Modify the program to find the trajectories of a positron in the electrostatic field due to a +1 point charge at (+.5,0,0) and a -1 point charge at (-.5,0,0).

These trajectories cannot be derived analytically. Other source charge distributions are considered in the problems at the end of the chapter.

THE MOTION OF CHARGED PARTICLES IN COMBINED ELECTROSTATIC AND MAGNETOSTATIC FIELDS

The algorithmic solution for charged particle trajectories becomes more difficult when the force is velocity-dependent. The program discussed above for electrostatic fields calculates the velocity a half-Δt time step away from the position and the acceleration. Since the acceleration needs the value of the velocity at the acceleration's own time, an estimate of this un-half-stepped velocity must be made. This estimated velocity can then be used to find the acceleration.

Example 2. Find the trajectory of a positron starting at (-1,0,0) with velocity (0,1,0) in a uniform magnetic field of magnitude (q/m) parallel to the z-axis.

The following listing is for a program which implements the algorithmic trajectory calculation for charges moving in combined electrostatic and magnetostatic fields.

```
MTONEB

100  READ E1,E2,E3,B1,B2,B3
110  DATA 0,0,0,0,0,1
120  PRINT "INITIAL X,Y,Z,VX,VY,VZ & FINAL T?"
130  INPUT X,Y,Z,V1,V2,V3,T0
140  LET D=.05
150  LET Q0=1
160  LET T=0
170  LET T9=0
180  LET A1=0
190  LET A2=0
200  LET A3=0
210  REM COMPUTE FIELDS, ACCELS., VEL., POSITION, TIME
220  LET U1=V1+A1*D/2
230  LET U2=V2+A2*D/2
240  LET U3=V3+A3*D/2
250  LET A1=Q0*(E1+(U2*B3-U3*B2))
260  LET A2=Q0*(E2+(U3*B1-U1*B3))
270  LET A3=Q0*(E3+(U1*B2-U2*B1))
280  IF T>0 THEN 330
290  LET V1=V1+A1*D/2
300  LET V2=V2+A2*D/2
310  LET V3=V3+A3*D/2
320  GOTO 360
330  LET V1=V1+A1*D
340  LET V2=V2+A2*D
350  LET V3=V3+A3*D
360  LET X=X+V1*D
370  LET Y=Y+V2*D
380  LET Z=Z+V3*D
390  LET T=T+D
400  LET T9=T9+D
410  IF ABS(T9-T0/10)/T9>.000001 THEN 440
420  LET T9=0
```

```
430  PRINT T,X,Y,Z
440. IF T<T0 THEN 210
450  PRINT
460  GOTO 120
470  END
```

Practice Exercise: Annotate the program by written comments next to each line.

A run of this program for the initial conditions given is:

```
RUN
MTONEB

INITIAL X,Y,Z,VX,VY,VZ & FINAL T?
?-1,0,0,0,1,0,10
 1.             -.539729         .841534         0
 2.              .417183         .907916         0
 3.              .989278         .137948         0
 4.00001         .64953         -.759143         0
 5.00001        -.28915         -.956982         0
 6.00001        -.962105        -.273279         0
 7.00002        -.749406         .662207         0
 8.00002         .153069         .987741         0
 9.00001         .914021         .403409         0
 9.99999         .832466        -.552571         0

INITIAL X,Y,Z,VX,VY,VZ & FINAL T?
?
```

The motion is a circle as is expected from the analytic solution. A sensitive test of the accuracy of this computer-based method is an examination of the magnitude of the velocity as the particle moves around the circle. The matter of accuracy is examined further in an exercise at the end of the chapter.

Motion in a magnetic field often involves all three directions, x, y and z. Hence, the motion is often hard to plot on a two-dimensional piece of paper.

Practice Exercise: Modify the program to compute trajectories when both a uniform electric field in the y direction and a uniform magnetic field in the z direction are present.

Such a field configuration is called a velocity selector. When the initial velocity of the charged particle has magnitude $|\vec{E}|/|\vec{B}|$ and is perpendicular to both fields, the trajectory is a straight line. Thompson used this field arrangement in his early measurement of (e/m) for electrons. When the velocity is not perpendicular to both fields, the motion can be quite complicated.

Practice Exercise: Modify the program to find the trajectories of a positron when both the electrostatic field due to a point charge at the origin and a uniform magnetic field in the z direction are present.

Analytical methods won't produce these trajectories even though the changes necessary in the algorithmic program are minimal. Further physical field situations are considered in the problems at the end of the chapter.

CONCLUSION

This chapter introduced a way to calculate the trajectories of charged particles in electrostatic and magnetostatic fields. The method works for arbitrary fields and for any charged particle. The method is general but easy to understand. By running several examples you can gain a good understanding of the motions of charged particles in e & m fields.

CHAPTER THREE EXERCISES

1. Consider the motion of a positron in the electric field created by an infinitely long line charge. The charge is along the z-axis and has a magnitude of +1 coulomb/m.

   a) *Determine the trajectory of a positron initially travelling in the z direction starting at (1,0,0) with a velocity of one.*

   b) *Determine the trajectory of a positron initially travelling with a velocity of (5,0,0) and an initial position of (-40,1,0).*

   c) *Does the angle of deflection of the positron change with the different y values for its starting point in part (b)? Try y=.5, y=2, and y=5.*

   d) *What happens when the charge on the line is changed to the charge on a (negative) electron?*

   e) *Will the electric field ever be able to hold the positron (i.e., will the positron ever orbit the line charge)?*

2. Consider the motion of an electron in the electric field created by two infinitely long line charges. The charges are parallel to the z-axis, each with a line charge density of -1. The line charges intersect the x-y plane at the points (0,2) and (0,-2).

   a) *Determine the trajectory of an electron initially travelling in the z direction starting at the origin with a velocity of one.*

   b) *Determine the trajectory of an electron initially travelling with a velocity of (5,0,0) and an initial position of (-20,0,0).*

   c) *Does the angle of deflection of the positron change with the different y values for its starting point in part (b)? Try y=-1, y=-.5, y=.5, y=1.*

   d) *What happens when the charge on the line is changed to the charge on one (negative) electron?*

3. Consider the motion of a positron in the electric field created by two point charges. The charges and positions are: −1 at (3,0,0) and −1 at (−3,0,0).

   a) *What happens to the positron with the initial conditions: position (0,8,0) and velocity (.45,0,0)?*

   b) *What happens to the positron with the initial conditions: position (4,0,0) and velocity (0,1,0)?*

   c) *What happens to the positron with the initial conditions: position (3,0,3) and velocity (0,.5,0)?*

   d) *Try other trajectories that you think might be interesting.*

4. Consider the motion of a positron in the electric field created by a dipole charge distribution. The charges and positions are: −1 at (3,0,0) and +1 at (−3,0,0).

   a) *What happens to the positron with the initial conditions: position (0,8,0) and velocity (.45,0,0)?*

   b) *What happens to the positron with the initial conditions: position (−4,0,0) and velocity (0,1,0)?*

   c) *What happens to the positron with the initial conditions: position (4,0,0) and velocity (0,1,0)?*

   d) *What happens to the positron with the initial conditions: position (3,0,3) and velocity (0,.5,0)?*

   e) *Try other trajectories you think might be interesting.*

5. Constant magnetic fields are sometimes used as simple electron beam collimators. Suppose we have a constant magnetic field in the z direction of magnitude (e/m).

   a) *What happens to an electron with initial position at the origin and initial velocity (0,0,1)?*

   b) *What happens to an electron with initial position at the origin and initial velocity (0,1,1).*

   c) *What happens to an electron with initial position at the origin and initial velocity (0,1,0).*

   d) *How does this field act as a simple collimator?*

6. Consider the diverging field $B_x = 0$, $B_y = 1/x^2$, and $B_z = 1/x^2$.

   a) *What happens to a positron with initial position of (20,0,0) and initial velocity of (−1,0,0). This is often called a magnetic mirror because the field pattern reflects charged particles. Two such mirrors contain charged particles between them and are often called magnetic bottles.*

b) *What happens to the velocity in the x direction?*

c) *Is energy conserved?*

7. Consider the combined electric and magnetic field problem discussed in the chapter (the velocity selector with a uniform $\vec{E}$ = q/m in the y direction and a uniform $\vec{B}$ = q/m in the z direction).

   a) *Verify that those particles that are perpendicular to both $\vec{E}$ and $\vec{B}$ and have a velocity of $|\vec{E}|/|\vec{B}|$ pass through this field as if the field did not exist.*

   b) *In what direction do the particles move if $|\vec{E}|/|\vec{B}|$ is greater than or less than the velocity?*

   c) *In what direction do the particles move if the x-component of velocity is $|\vec{E}|/|\vec{B}|$ but the electron has some velocity in the y direction?*

   d) *In what direction do the particles move if the x-component of velocity is $|\vec{E}|/|\vec{B}|$ but the electron has some velocity in the z direction?*

   e) *Determine analytically that $V_x = |\vec{E}|/|\vec{B}|$ for the velocity selector.*

8. Consider a uniform magnetic field of .1 in the z direction and a point charge of +1 at the origin.

   a) *If an electron has an initial position of (1,0,0) and velocity of (0,1,0), what does its trajectory look like?*

   b) *If an electron has an initial position of (1,0,0) and velocity of (0,0,1), what does its trajectory look like?*

9. The uniform electric field problem is similar to the acceleration of a body under the influence of gravity.

   a) *Write down the analytical expression in component form for the position of a particle as a function of time (including its initial velocity).*

   b) *Compare the results of the program MTONE to the analytical expression at the times t = 2, 5, and 10.*

10. The solution to the problem of a positron moving in a constant magnetic field is a good problem to check the iterative solution because there is an analytical solution. Start the charged particle with initial velocity perpendicular to the magnetic field.

    a) *Determine (analytically) the radius of the (circular) orbit as a function of the charge, mass, magnetic field, and velocity.*

b) Determine the period of the motion as a function of the magnetic field, charge, and mass. (The frequency associated with this period is called the cyclotron frequency.)

c) Compare your results with that obtained by the algorithmic method.

d) The analytical solution has $|\vec{V}|$ = constant. Check your iterative solution by examining how constant $|\vec{V}|$ remains as time passes.

CHAPTER FOUR: GAUSS'S, AMPERE'S, AND FARADAY'S LAWS FOR STATIC FIELDS

INTRODUCTION

Gauss's, Ampere's and Faraday's Laws represent the original forms of Maxwell's four equations. Aside from the term Maxwell added to Ampere's Law (which made the set of equations agree with conservation of charge and also predict electromagnetic waves), these four equations represent the basis of all classical electricity and magnetism.

This chapter shows you a way to use the computer to calculate the integrals in Gauss's Laws for $\vec{E}$ and for $\vec{B}$, Ampere's Law and Faraday's Law. You can then deal with questions such as "Can this given field be an electrostatic field? If so, where are the charges and what values of charge are present?" or "Can this given field be a magnetostatic field, and, if so, where and of what size are the currents?" (Again we will use the term magnetostatic field for $\vec{B}$ even though magnetostatic induction is strictly correct.) In the forms we shall use, the laws can be written as follows:

Gauss's Law for $\vec{E}$	$\oint \vec{E} \cdot d\vec{A} = 4\pi k_0 q$	(11)
Gauss's Law for $\vec{B}$	$\oint \vec{B} \cdot d\vec{A} = 0$	(12)
Ampere's Law	$\oint \vec{B} \cdot d\vec{\ell} = k_0 I$	(13)
Faraday's Law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	(14)

(In the general case Faraday's Law says that $\oint \vec{E} \cdot d\vec{\ell} = -d\Phi/dt$, but we are dealing with static cases. Currents are constant in time so the flux, Φ does not change.) The symbol $\oint$ means the closed integral: in Gauss's Laws the closed integral is taken over some surface completely enclosing some volume of space; in Ampere's and Faraday's Laws the closed integral is taken along some contour completely enclosing a surface in space. k_0 defines the units, and q and I are the net enclosed charge and current, respectively.

These laws give a way to examine any field to see if it can represent an electrostatic field $\vec{E}$ (in which case $\oint \vec{E} \cdot d\vec{\ell}$ must be zero everywhere and $\oint \vec{E} \cdot d\vec{A}$ can tell you the values of the charges present) or a magnetostatic field $\vec{B}$ (in which case $\oint \vec{B} \cdot d\vec{A}$ must be zero everywhere and $\oint \vec{B} \cdot d\vec{\ell}$ can tell you the values of the currents present). These integral forms of the four basic laws also determine the four differential forms for Maxwell's equations which use vector derivatives (the divergence and the curl).

The chapter starts by presenting a number of pictures representing possible vector fields. These pictures will be examined to see which might represent electrostatic or magnetostatic fields. The chapter will then move on to show you how the computer can be used to deal more quantitatively with fields using the four basic laws in their integral forms. The most general question of how to deal with fields if you are given only a set of measured values throughout some region of space is considered in the exercises at

the end of the chapter. The discussion in the chapter will be given in terms of $\vec{E}$ and $\vec{B}$ fields, but the methods are applicable to any vector field (such as the gravitational field or the velocity field of a flow pattern).

PICTURES OF VECTOR FIELDS

It is sometimes helpful to see a number of pictures of possible vector fields. Inspecting the pictures to determine whether or not each picture might represent an electrostatic or magnetostatic (or gravitational or whatever) field exercises your knowledge and comprehension of the basic laws. By inspection, you can often guess whether or not a particular picture could represent an $\vec{E}$ or $\vec{B}$ field; you can often guess where the source charges or source currents would lie. To be more quantitative, say to find the values of the charges or currents, you need to perform the necessary integrations. The computer helps in two ways: first, the computer can be used to produce the pictures of the fields, and second, the computer can perform the integrations.

There are two common kinds of pictures which are used to represent vector fields for discussion. The first is a figure that shows the vector field represented as arrows at a number of points throughout a region. The second is a field line map such as those discussed in Chapters One and Two. We will concentrate on pictures of the first kind. Let us look at several examples:

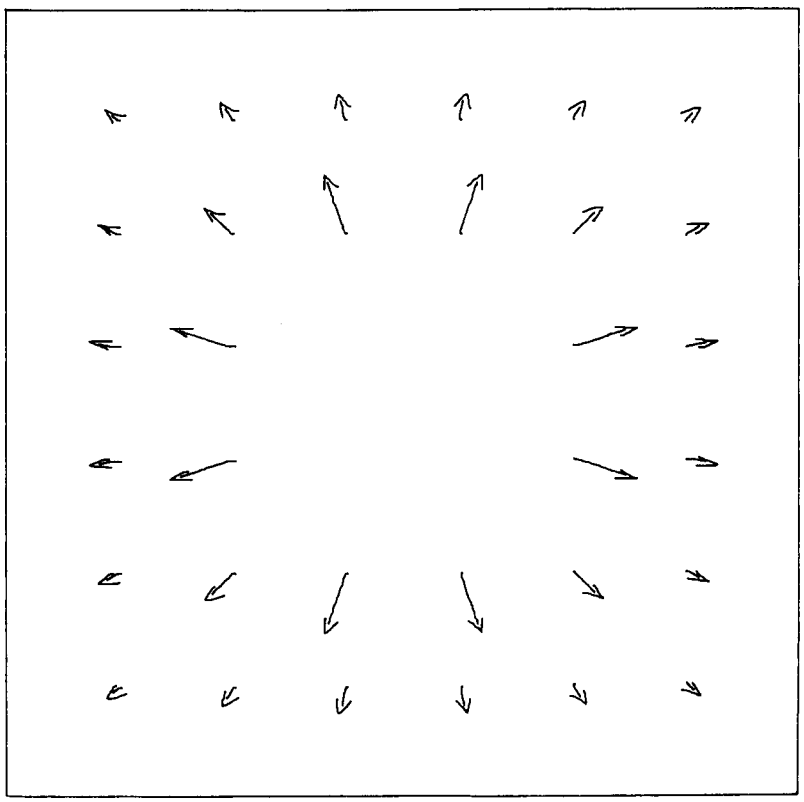


Figure (a)

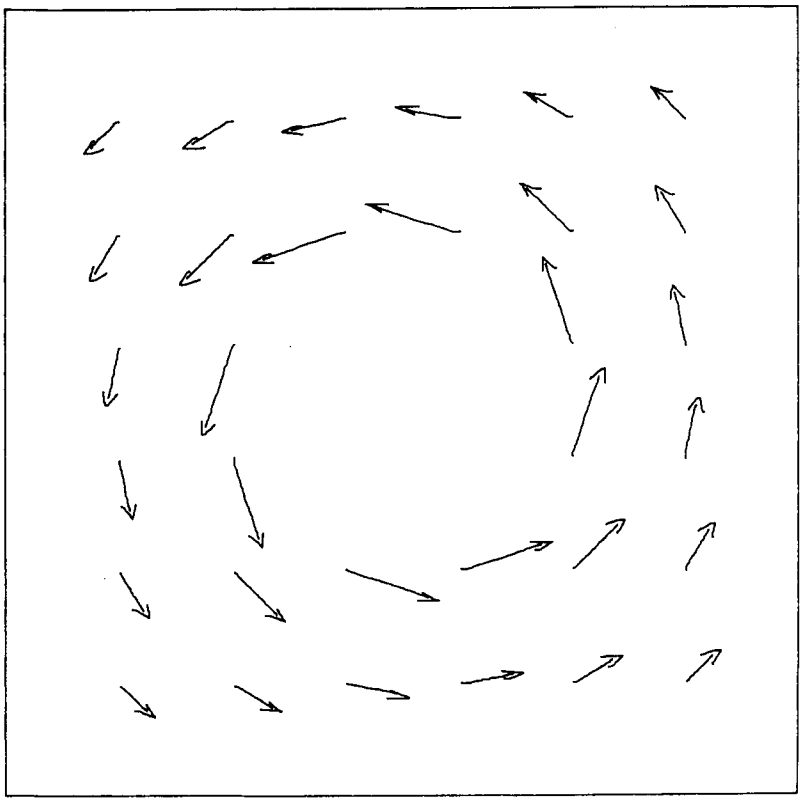


Figure (b)

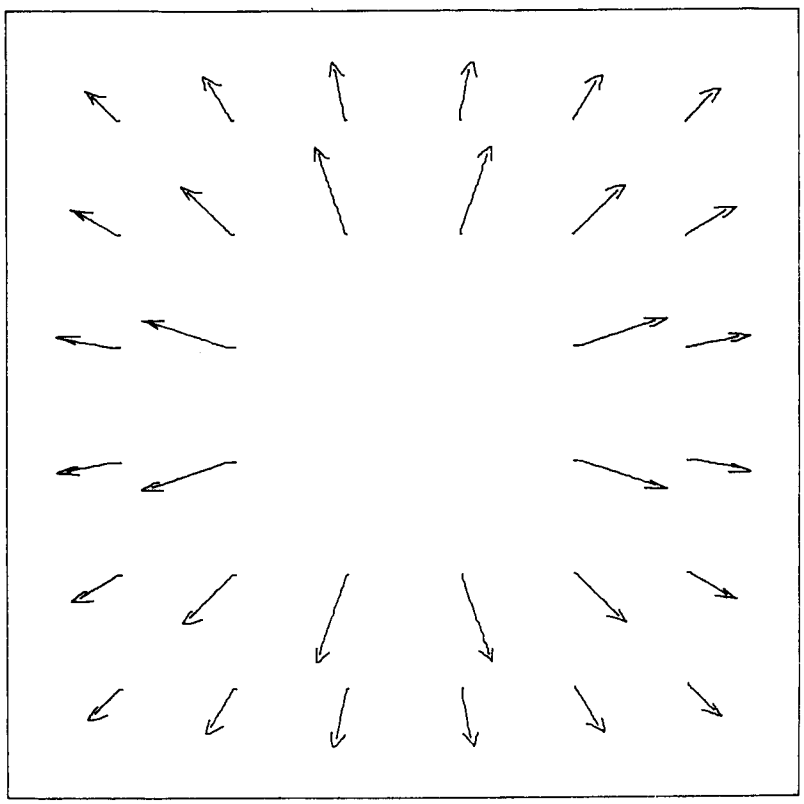


Figure (c)

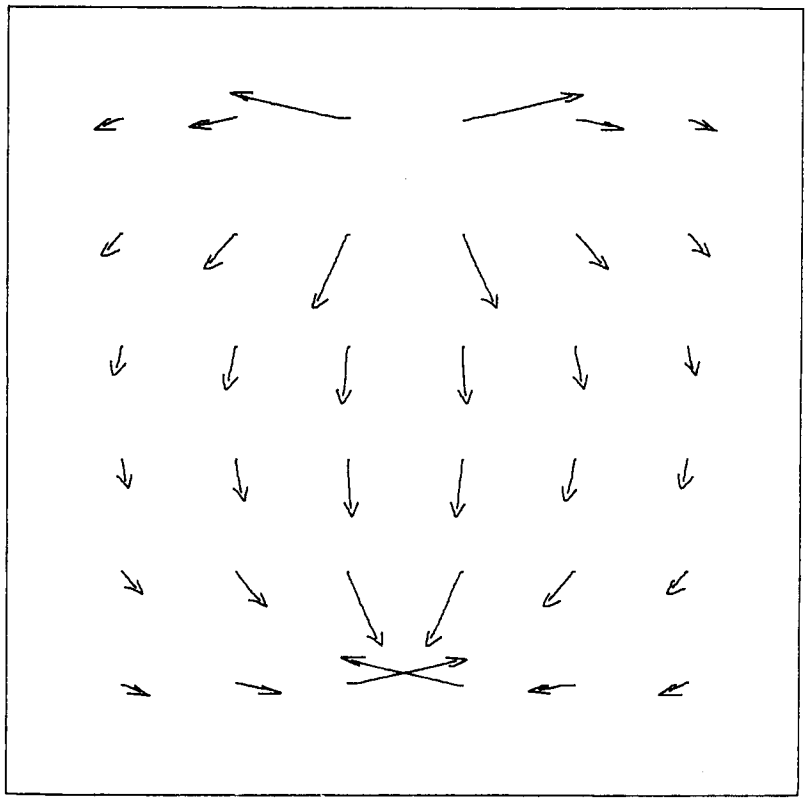


Figure (d)

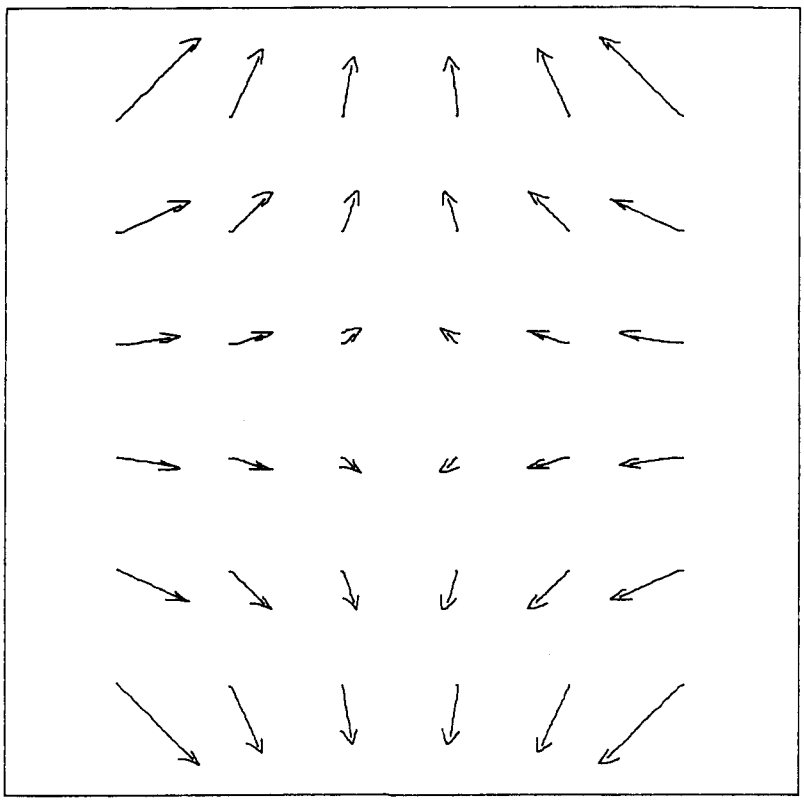


Figure (e)

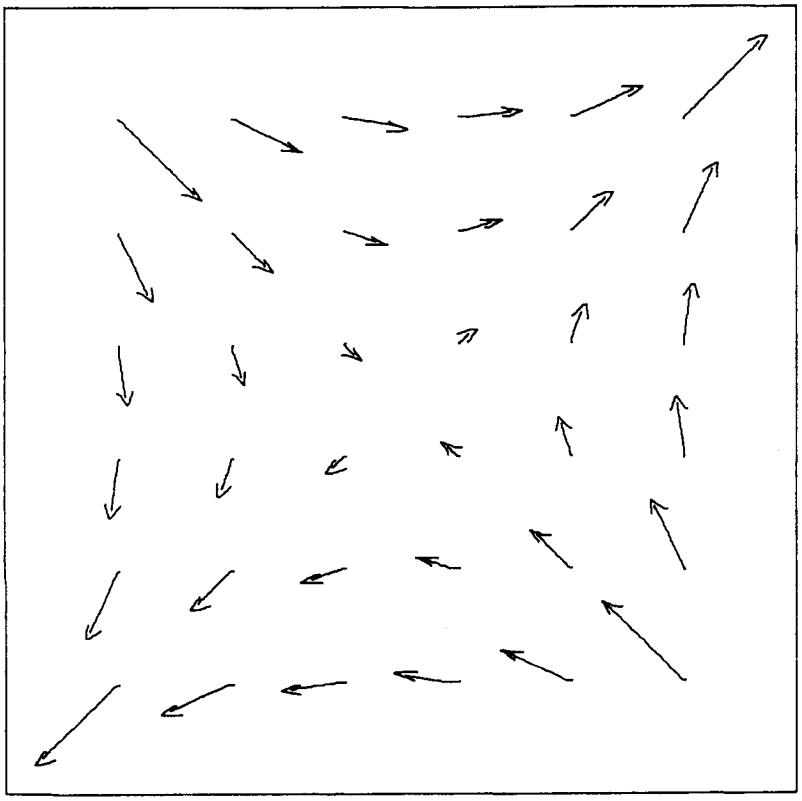


Figure (f)

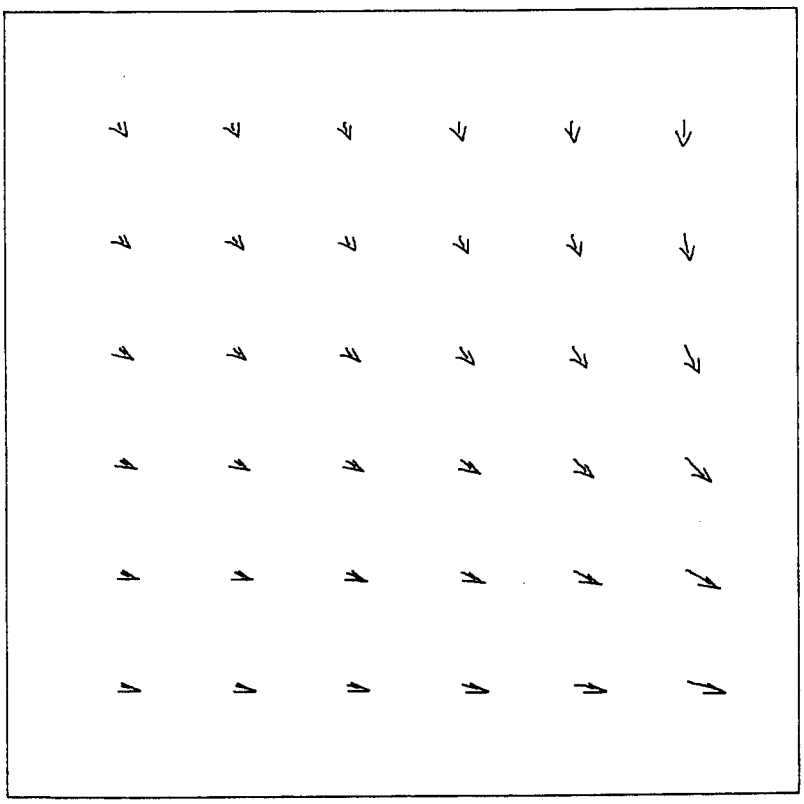


Figure (g)

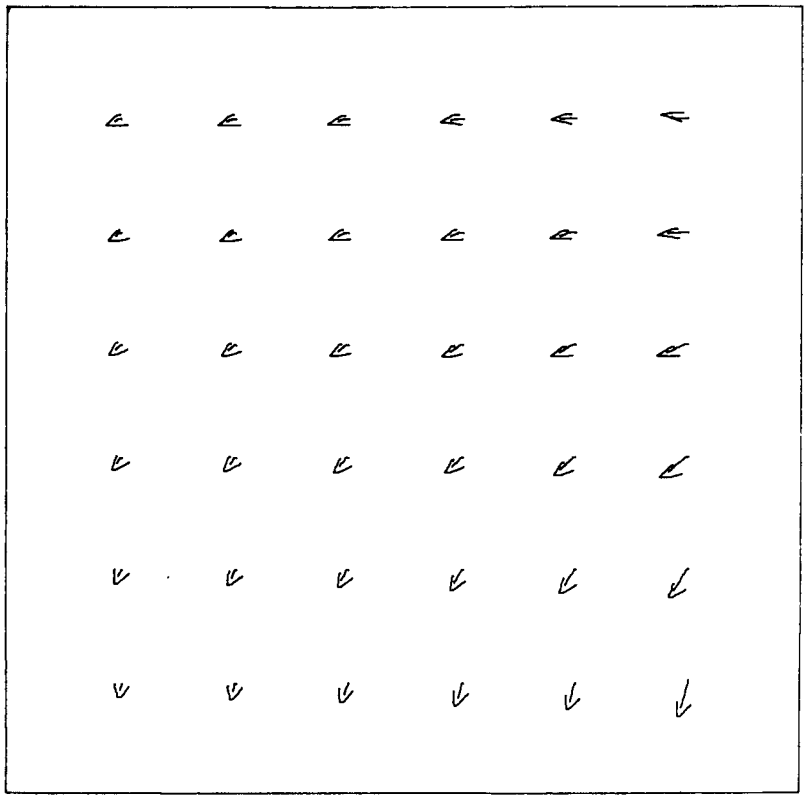


Figure (h)

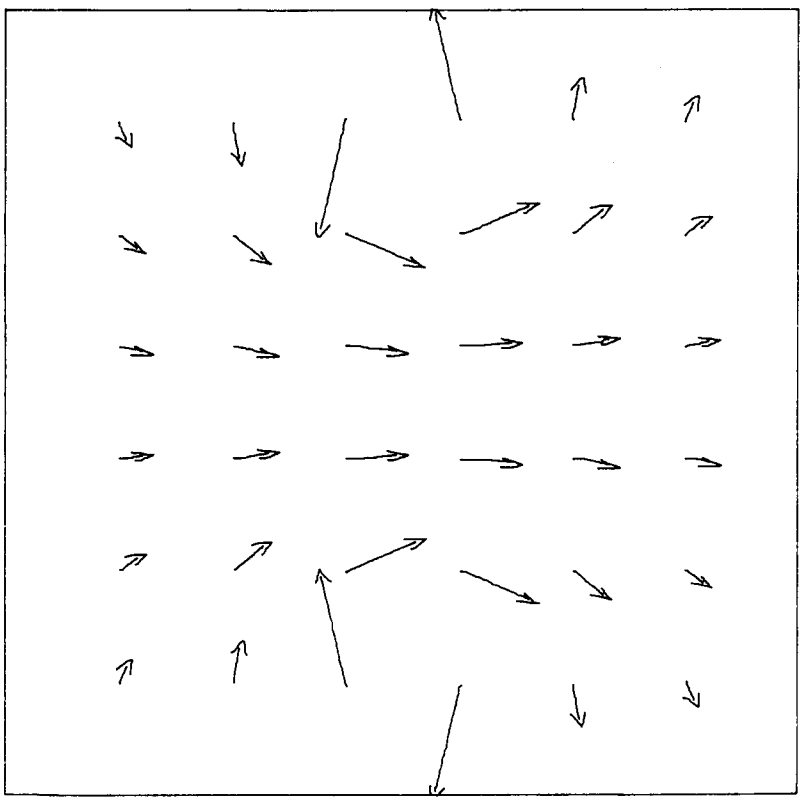


Figure (i)

Which of these figures could represent an electrostatic field? You need to identify those fields for which $\oint \vec{E} \cdot d\vec{\ell} = 0$ for all (closed) paths. Figure (a) is such a field; it appears possible that adding up $\vec{E} \cdot \Delta\vec{\ell}$ for small pieces around a contour made up of radial and circumferential pieces would result in zero. In principle, you would have to show that the integral was zero for any possible contour; in practice, if one type of contour has $\oint \vec{E} \cdot d\vec{\ell} = 0$, it is likely that almost any contour will also have $\oint \vec{E} \cdot d\vec{\ell} = 0$. In the case of Figure (a), it seems clear that any contour which can be approximated by a set of radial and circumferential pieces might obey Faraday's Law ($\oint \vec{E} \cdot d\vec{\ell} = 0$), and that is certainly a wide class of contours.

On those figures you believe could be electrostatic fields, identify the positions and relative values of the source charges. In Figure (a) there seems to be a positive charge at the center of the figure (and an equal amount of negative charge at ∞ on which to terminate the field lines). When several charges are present, you can estimate the relative magnitudes and the signs of the charges by considering the field near each charge.

Which of these figures could represent a magnetostatic field? This is a harder question because you really need to find fields for which $\oint \vec{B} \cdot d\vec{A} = 0$ and the figures are only two-dimensional. Let us agree that the physical situations represented in the figures have no dependence upon z, that is, that every plane parallel to the x-y plane is equivalent to the x-y plane. Then the x-y plane shown in the figures tells the whole story. Figure (b) seems likely to have $\oint \vec{B} \cdot d\vec{A} = 0$ for surfaces which are pieces of circular cylinders with z-axis axes connected by radial planes. Again you cannot prove that $\oint \vec{B} \cdot d\vec{A} = 0$ for all possible surfaces without doing some integrals, but the result looks right.

On those figures you believe could be magnetostatic fields, identify the positions and relative values of the source currents. In figure (b) there seems to be only one source current at the center of the picture. The current appears to come out of the page (by the right-hand rule), so the current is positive although its magnitude cannot be determined without more information. If there were several currents, their relative values could be estimated from the relative strengths of the fields nearby; the directions of the currents could be determined from the directions of the fields nearby.

Now you can try the rest of the figures. It is not difficult for anyone to produce pictures such as those shown. You can do it yourself.

Practice Exercise: Write a program to calculate $|\vec{E}|$ and the angle $\vec{E}$ makes with the x-axis for a number of points on the x-y plane. You may assume E_x, E_y, E_z are given as equations.

SURFACE AND LINE INTEGRALS WITH THE COMPUTER

Now let us see how to be more quantitative. It is easy to do the integrals involved in Gauss's, Ampere's and Faraday's Laws on the computer. An exact integration would have to be analytical, but most field patterns cannot be integrated analytically. The method of integration we will present is approximate (although the method can be made accurate as desired) but it can integrate the necessary surface and line integrals for any vector field.

To perform the line integral, you break the contour up into small segments, $\Delta\vec{\ell}$, evaluate the field, $\vec{F}$, at the center of each segment, and form the dot-products $\vec{F}\cdot\Delta\vec{\ell}$. Adding up the terms $\vec{F}\cdot\Delta\vec{\ell}$ for all the segments that form a closed loop yields an approximation to $\oint\vec{F}\cdot d\vec{\ell}$ for the vector field $\vec{F}$. Again, in the limit of the sum of infinitesimal segments, $d\vec{\ell}$, the result would be exact.

To perform the surface integral, you break the surface up into small areas, $\Delta\vec{A}$, evaluate the field, $\vec{F}$, at the center of each small area, and form the dot-products $\vec{F}\cdot\Delta\vec{A}$. Adding up the terms $\vec{F}\cdot\Delta\vec{A}$ for all the small areas forming a closed surface is then an approximation to the integral $\oint\vec{F}\cdot d\vec{A}$ for the vector field $\vec{F}$. In the limit of the sum infinite simal areas, $d\vec{A}$, is by definition the integral; for finite $\Delta\vec{A}$, the sum is an approximation.

Example 1. Evaluate $\oint\vec{F}\cdot d\vec{A}$ for the vector electrostatic field, $\vec{F}$, due to a line charge coinciding with the z-axis. Consider rectangular parallelepiped surfaces.

A program to find this surface integral might look like the following:

```
GAUSS

100  PRINT "(X,Y,Z) FOR 2 CORNERS?"          ] Set region
110  INPUT X1,Y1,Z1,X2,Y2,Z2                 ]
120  LET P0=0 - Flux
130  LET N0=16 - # of segments in x, y and z
140  LET K0=1 - Units
150  LET Q=+1 - Magnitude of charge
160  LET X0=(X2-X1)/N0 - Δx
170  LET Y0=(Y2-Y1)/N0 - Δy
180  LET Z0=(Z2-Z1)/N0 - Δz
190  LET A0=X0*Y0 - ΔA
200  FOR X=X1+X0/2 TO X2-X0/2 STEP X0        ]
210  FOR Y=Y1+Y0/2 TO Y2-Y0/2 STEP Y0        ]
220  LET F3=K0*Q*Z2/(X*X+Y*Y+Z2*Z2)          ]
230  LET P0=P0+F3*A0                         ] Two X-Y
240  LET F3=K0*Q*Z1/(X*X+Y*Y+Z2*Z2)          ] Planes at
250  LET P0=P0-F3*A0                         ] Z1 and Z2
260  NEXT Y                                  ]
270  NEXT X                                  ]
280  LET A0=Y0*Z0-ΔA
290  FOR Y=Y1+Y0/2 TO Y2-Y0/2 STEP Y0        ]
300  FOR Z=Z1+Z0/2 TO Z2-Z0/2 STEP Z0        ]
310  LET F1=K0*Q*X2/(X2*X2+Y*Y+Z*Z)          ]
320  LET P0=P0+F1*A0                         ] Two Y-Z
330  LET F1=K0*Q*X1/(X1*X1+Y*Y+Z*Z)          ] Planes
340  LET P0=P0-F1*A0                         ]
350  NEXT Z                                  ]
360  NEXT Y                                  ]
370  LET A0=X0*Z0-ΔA
380  FOR X=X1+X0/2 TO X2-X0/2 STEP X0        ]
390  FOR Z=Z1+Z0/2 TO Z2-Z0/2 STEP Z0        ]
400  LET F2=K0*Q*Y2/(X*X+Y2*Y2+Z*Z)          ]
410  LET P0=P0+F2*A0                         ]
420  LET F2=K0*Q*Y1/(X*X+Y1*Y1+Z*Z)          ] Two X-Z
430  LET P0=P0-F2*A0                         ] Planes
440  NEXT Z                                  ]
450  NEXT X                                  ]
460  PRINT "FLUX =";P0                       ]
470  PRINT
480  GOTO 100
490  END
```

The field, $\vec{F}$ = (F1,F2,F3), is the electrostatic field due to a line charge at the origin $[k_0\lambda/r(\vec{r}/|\vec{r}|)]$, so $\oint\vec{F}\cdot d\vec{A}$ should be zero unless the region enclosed by the rectangular parallelepiped encloses the z-axis. When the z-axis is enclosed, $\oint\vec{F}\cdot d\vec{A}$ should equal $4\pi k_0\lambda(Z2\text{-}Z1)$. A run of this Gauss's Law program looks like:

```
RUN
GAUSS

(X,Y,Z) FOR 2 CORNERS?
?-1,-1,-1,1,1,1
FLUX = 15.3603

(X,Y,Z) FOR 2 CORNERS?
?1,1,1,2,2,2
FLUX = .217896

(X,Y,Z) FOR 2 CORNERS?
?1,1,1,2,2,3
FLUX = .355734

(X,Y,Z) FOR 2 CORNERS?
?
```

Practice Exercise: *How much charge lies along the line charge between Z = 0 and z = 2? What is the linear charge density, λ? Does this value agree with the equations in the program?*

Practice Exercise: *Modify the program for line charges having $\lambda = +1$ and -1 which are parallel to the z-axis and cut the x-y plane at (+.5,0) and (−.5,0), respectively. Where should the surface integral be zero and where non-zero?*

Practice Exercise: *Modify the program for a long, straight wire carrying current I which coincides with the z-axis. (Remember that $\vec{B}$ is perpendicular to $\vec{r}$.) Where should the surface integral be zero and where non-zero?*

Further examples are considered in the exercises at the end of the chapter. The field, $\vec{F}$, can be given as equations or even as data.

Example 2. Evaluate $\oint\vec{F}\cdot d\vec{\ell}$ for the vector magnetostatic field due to a long, straight wire carrying a current of 1 ampere and coinciding with the z-axis. Consider rectangular contours lying in the x-y plane.

A program to evaluate the line integral might look like:

```
AMPFAR

100  PRINT "(X,Y) FOR 2 CORNERS?"
110  INPUT X1,Y1,X2,Y2
120  LET L0=0
130  LET N0=512
140  LET K0=1.E-07
150  LET I=1
160  LET X0=(X2-X1)/N0
170  LET Y0=(Y2-Y1)/N0
180  FOR X=X1+X0/2 TO X2-X0/2 STEP X0
190  LET F=-2*K0*I*Y1/(X*X+Y1*Y1)
200  LET L0=L0+F*X0
210  LET F=-2*K0*I*Y2/(X*X+Y2*Y2)
220  LET L0=L0-F*X0
230  NEXT X
240  FOR Y=Y1+Y0/2 TO Y2-Y0/2 STEP Y0
250  LET F=2*K0*I*X1/(X1*X1+Y*Y)
260  LET L0=L0-F*Y0
270  LET F=2*K0*I*X2/(X2*X2+Y*Y)
280  LET L0=L0+F*Y0
290  NEXT Y
300  PRINT "LINE INTEGRAL =";L0
310  PRINT
320  GOTO 100
330  END
```

Since $\vec{F}$ = (F1,F2,F3) is due to a long straight wire ($\mu_0 I/2\pi r$, tangential with sense by the right-hand rule), you expect $\oint \vec{F} \circ d\vec{\ell}$ to equal zero unless the rectangular contour encloses the z-axis. When the contour encloses the z-axis, you expect $\oint \vec{F} \cdot d\vec{\ell} = \mu_0 I$ in mks units. A run of this line integral program looks like:

```
RUN
AMPFAR

(X,Y) FOR 2 CORNERS?
?-1,-1,1,1
LINE INTEGRAL = 1.25664E-06

(X,Y) FOR 2 CORNERS?
?1,1,2,2
LINE INTEGRAL =-3.77476E-15

(X,Y) FOR 2 CORNERS?
?
```

Practice Exercise: Modify the program for two, long straight wires parallel to the z-axis, carrying currents of +1 and −1, and cutting the x-y plane at (+.5,0) and (−.5,0), respectively. Where should the line integral be zero and where non-zero?

Practice Exercise: Modify the program for a single line charge coinciding with the z-axis and having a linear charge density, λ, of +1. Where should the line integral be zero and where non-zero?

Further examples are considered in the problems at the end of the chapter. The vector field $\vec{F}$ could be given as equations or as data.

CONCLUSION

This chapter has displayed pictures of several vector fields. You were asked to try to identify which pictures could represent electrostatic and magnetostatic fields. The chapter then showed you a way to compute the surface and line integrals necessary for Gauss's, Ampere's and Faraday's Laws in order to check vector fields quantitatively. By working a few examples from the exercises, you can deepen your understanding further.

CHAPTER FOUR EXERCISES

1. Consider the field distribution created by four line charges parallel to the z-axis. The line charge densities are: +1 at (1,1), +1 at (1, −1), −1 at (−1,1) and −1 at (−1,−1).

   a) Consider the region around the (1,1) line. What is the surface integral? Is it dependent on the size of the box around the charge? If so, is it dependent on all lengths or just one? What is the line integral around this point ($\oint \vec{F} \cdot d\vec{\ell}$)? Is it dependent on the size of the contour?

   b) Consider the region of space for positive x that includes both the lines at (1,1) and (1,−1). What is the surface integral ($\oint \vec{F} \cdot d\vec{A}$)? Is it dependent on the size of the box? If so, on what does it depend? What is the line integral ($\oint \vec{F} \cdot d\vec{\ell}$) around this region? Is it dependent on the size of the contour?

   c) Consider the region including all the line charges. What is the surface integral ($\oint \vec{F} \cdot d\vec{A}$)? Is it dependent on the size of the box? If so, how is it dependent? What is the line integral ($\oint \vec{F} \cdot d\vec{\ell}$) around the region? Is it dependent on the size of the contour?

   d) Consider the region inside the four points. What is the surface integral ($\oint \vec{F} \cdot d\vec{A}$)? What is the line integral ($\oint \vec{F} \cdot d\vec{\ell}$)?

   e) Do your results agree with Gauss's and Faraday's Laws?

2. Consider the magnetostatic field distribution created by four current lines parallel to the z-axis. The positions and currents are: +1 at (1,1), +1 at (1,−1), −1 at (−1,1) and −1 at (−1,−1).

   a) Consider the region around the (1,1) line. What is the surface integral ($\oint \vec{F} \cdot d\vec{A}$)? Is it dependent on the size of the box around the charge? If so, is it dependent on all lengths or just one? What is the line integral around this point ($\oint \vec{F} \cdot d\vec{\ell}$)? Is it dependent on the size of the contour?

   b) Consider the region of space for positive x that includes both the lines at (1,1) and (1,−1). What is the surface integral ($\oint \vec{F} \cdot d\vec{A}$)? Is it dependent on the size of the box? If so, on what does it depend? What is the line integral ($\oint \vec{F} \cdot d\vec{\ell}$) around this region? Is it dependent on the size of the contour?

   c) Consider the region including all the line charges. What is the surface integral ($\oint \vec{F} \cdot d\vec{A}$)? Is it dependent on the size of the box? If so, how is it dependent? What is the line integral ($\oint \vec{F} \cdot d\vec{\ell}$) around the region? Is it dependent on the size of the contour?

   d) Consider the region inside the four points. What is the surface integral ($\oint \vec{F} \cdot d\vec{A}$)? What is the line integral ($\oint \vec{F} \cdot d\vec{\ell}$)?

   e) Do your results agree with Gauss's and Faraday's Laws?

3. Consider a combined electrostatic and magnetostatic field distribution. Part of it is created by two line charges parallel to the z-axis: +3 linear charge density at (1,1) and −3 linear charge density at (−1,1). The other part is created by two line currents parallel to the z-axis with currents and positions: +2 at (1,−1) and −2 at (−1,−1).

   a) *Consider the region around the (1,1) line. What is the surface integral* $(\oint \vec{F} \cdot d\vec{A})$*? Is it dependent on the size of the box around the charge? If so, is it dependent on all lengths or just one? What is the line integral around this point* $(\oint \vec{F} \cdot d\vec{\ell})$*? Is it dependent on the size of the contour?*

   b) *Consider the region of space for positive x that includes both the lines at (1,1) and (1,−1). What is the surface integral* $(\oint \vec{F} \cdot d\vec{A})$*? Is it dependent on the size of the box? If so, on what does it depend? What is the line integral* $(\oint \vec{F} \cdot d\vec{\ell})$ *around this region? Is it dependent on the size of the contour?*

   c) *Consider the region including all the line charges. What is the surface integral* $(\oint \vec{F} \cdot d\vec{A})$*? Is it dependent on the size of the box? If so, how is it dependent? What is the line integral* $(\oint \vec{F} \cdot d\vec{\ell})$ *around the region? Is it dependent on the size of the contour?*

   d) *Consider the region inside the four points. What is the surface integral* $(\oint \vec{F} \cdot d\vec{A})$*? What is the line integral* $(\oint \vec{F} \cdot d\vec{\ell})$*?*

   e) *Do your results agree with Gauss's and Faraday's Laws?*

4. Consider the following field.

$$F_x = 0$$

$$F_y = 0$$

$$\begin{cases} F_z = 1 \text{ for } z > 0 \\ F_z = 0 \text{ for } z = 0 \\ F_z = -1 \text{ for } z < 0 \end{cases}$$

   a) *What is the surface integral for the box defined by (−1,−1,−1) and (1,1,1)? If it is not zero, is the surface integral dependent on the size of the box centered at (0,0,0)? In what way?*

   b) *If the surface integral is not zero, can you identify the charge distribution that creates this field?*

   c) *What are the line integrals for the square loops defined by (1,1,0) and (−1,−1,0); (1,0,1) and (−1,0,−1); (0,1,1) and (0,−1,−1).*

   d) *If the line integral is not zero, can you identify the current distribution that creates this field?*

5. Consider the following field.

$$F_x = 0$$

$$F_y = 0$$

$$\begin{cases} F_z = 1 \text{ for } x > 0 \\ F_z = 0 \text{ for } x = 0 \\ F_z = -1 \text{ for } x < 0 \end{cases}$$

a) What is the surface integral for the box defined by (−1,−1,−1) and (1,1,1)? If it is not zero, is the surface integral dependent on the size of the box centered at (0,0,0)? In what way?

b) If the surface integral is not zero, can you identify the charge distribution that creates this field?

c) What are the line integrals for the square loops defined by (1,1,0) and (−1,−1,0); (1,0,1) and (−1,0,−1); (0,1,1) and (0,−1,−1).

d) If the line integral is not zero, can you identify the current distribution that creates this field?

6. Consider the following field:

$$\vec{F} = \hat{r} * r(2+r^2)$$

a) What is the surface integral for the box defined by (−1,−1,−1) and (1,1,1)? If it is non-zero, is the surface integral dependent on the size of the box centered at (0,0,0)? In what way?

b) If the surface integral is not zero, can you identify the charge distribution that creates this field?

c) Can you prove that the line integral for any field that can be written as $\vec{F} = \hat{r}f(r)$ is zero?

7. Not all fields are electric or magnetic. For example, it is often worthwhile to consider velocity fields of fluids. Consider the following fluid velocity field:

$$V_z = 0$$

$$V_y = 0$$

$$V_x = (1 - y^2/4)$$

a) What is the surface integral for the box defined by (−1,−1,−1) and (1,1,1)? If it is non-zero, is the surface integral dependent on the size of the box centered at (0,0,0)? In what way?

b) What does this surface integral mean physically (i.e., are there sources or sinks for the fluid within the region)?

c) What are the line integrals for the square loops defined by (1,1,0) and (−1,−1,0); (1,0,1) and (−1,0,−1); (0,1,1) and (0,−1,−1)?

d) What is the line integral around the square loop defined by (0,0,0) and (1,1,0)?

CHAPTER FIVE: THE LAPLACE AND POISSON EQUATIONS

INTRODUCTION

A combination of Laplace's and Poisson's equations represents one of the fundamental ways to describe electrostatic potentials. (Similar equations arise in many other branches of e & m and of physics in general.) You sometimes know the potential, V, everywhere on some surface that encloses a definite volume, and you want to find the potential throughout that volume. If the volume has a known volume charge density, $\rho(x,y,z)$ (measured as charge per unit volume), then you can calculate the potential everywhere using Poisson's equation. If the volume charge density, ρ is zero through the region, then Poisson's equation reduces to Laplace's equation, one of the simplest partial differential equations used commonly in physics.

Poisson's equation in three dimensions is

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=4\pi k_0\rho(x,y,z) \qquad (15)$$

When $\rho(x,y,z) = 0$ you have Laplace's equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}\equiv 0 \qquad (16)$$

Derivations of these equations and discussions of their analytical solutions can be found in standard physics textbooks.

These equations can be solved easily on a computer, and they represent simple examples of how partial differential equations can be solved numerically. For convenience in displaying results, we will consider two dimensional cases involving only x and y. The method of solution is valid for three dimensions, too, but more complicated partial differential equations sometimes demand more sophisticated methods of numerical solution than the method we will present.

DIFFERENCE EQUATIONS

The simplest method for solving partial differential equations numerically is based on replacing the differential equation (involving infinitesimal dx, dy, dz) by a difference equation (involving finite differences Δx, Δy, Δz). There are three criteria that the resulting difference equation must satisfy: 1) it must represent the partial differential equation correctly in the limit as $\Delta x\rightarrow 0$, $\Delta y\rightarrow 0$ and $\Delta z\rightarrow 0$; 2) it must be stable (which means that, if a small error occurs at some numerical step, the error must not grow); and 3) it must converge to some value (in other words, it must produce an answer). Laplace's equation is such that a simple difference equation approach can be shown to satisfy all three requirements. If the volume charge density, ρ, does not vary too wildly in space, the method will also work for Poisson's equation.

We start by defining a grid of points, labelled by i and j (for our two dimensional problems), covering the region where we want to compute the potential, V. How would the first partial derivative, $\partial V/\partial x$, be written in terms of the values of $V_{i,j}$ on the grid points? You could use the forward difference expression for a derivative $\partial V/\partial x \approx (V_{i+1,j} - V_{i,j})/\Delta x$ or you could use the corresponding backward derivative. A better approximation is called the central difference approximation and uses grid points in front of and behind the grid point (i,j) in question. Then $\partial V/\partial x \approx (V_{i+1,j} - V_{i-1,j})/(2\Delta x)$.

The second partial derivative, $\partial^2 V/\partial x^2$, equals $\partial(\partial V/\partial x)/\partial x$ so we can apply the same idea of approximating the partial derivatives by difference quotients. Using the central difference approximation, $\partial^2 V/\partial x^2 \approx (V_{i+1,j} - 2V_{i,j} + V_{i-1,j})/(\Delta x)^2$.

Practice Exercise: Show that the difference equation approximating $\partial^2 V/\partial x^2$ comes from two applications of the central difference approximation for the first derivative. The first application gives the first derivatives at (i+.5,j) and (i−.5,j); the second application yields the second derivative at (i,j).

THE DIFFERENCE EQUATIONS FOR LAPLACE'S AND POISSON'S EQUATIONS

Using the expression for approximations to the second partial derivatives of V on the grid of points, the (two-dimensional) Laplace's equation becomes

$$V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j} = 0 \qquad (17)$$

assuming $\Delta x = \Delta y$.

Practice Exercise: Derive this difference equation for Laplace's equation.

Practice Exercise: Derive the three-dimensional difference equation for Laplace's equation.

This difference equation can be solved for the potential at the grid point (i,j):

$$V_{i,j} = (V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1})/4 \qquad (18)$$

which says the potential at (i,j) is the average of the potentials at the nearest neighbor grid points. This equation for V suggests a simple strategy on the computer: Start with any values of the potential inside the region and with the known values on the grid points on the surface; moving sequentially through the interior grid points, replace the inside values by the average of nearest neighbor values; then repeat the process until interior values of the potential stop changing. The boundary values, which remain constant, will gradually influence the values throughout the interior. When all the inside values become equal to the average of their nearest neighbors (that is,after

all the values of the potential at interior points stop changing), then you have found the solution of Laplace's equation with the given potentials on the surface.

Poisson's equation is just as simple. The difference equation (in two dimensions) becomes

$$(V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j}) = R_{i,j}\,(\Delta S)^2 \qquad (19)$$

where $\Delta x = \Delta y = \Delta S$ and where $R_{i,j}$ is just the (known) quantity, $4\pi k_0 \rho(x,y)$, at the grid point (i,j). k_0 sets the units. Since $R_{i,j}$ does not change value at each particular (i,j) as you iterate over the grid, programs solving Poisson's equation by the difference equation approach look very similar to those solving Laplace's equation.

Example 1. Find the potential V in a charge-free, two-dimensional, square region given that the potential varies linearly around the edge from V = 0 at one corner to V = 18 at the opposite corner.

A program implementing the iterative strategy to solve Laplace's equation for this problem might look like the following:

```
LAPLAC

100  DIM V(10,10),U(10,10)
110  READ V(1,1),V(10,1),V(10,10),V(1,10)
120  DATA 0,9,18,9
130  LET V0=V(1,1)+V(1,10)+V(10,10)+V(10,1)
140  FOR I=2 TO 9
150  LET V(1,I)=I-1
160  LET V(10,I)=9+I-1
170  LET V(I,1)=I-1
180  LET V(I,10)=9+I-1
190  LET V0=V0+V(1,I)+V(10,I)+V(I,1)+V(I,10)
200  NEXT I
210  FOR I=2 TO 9
220  FOR J=2 TO 9
230  LET V(I,J)=V0/36
240  NEXT J
250  NEXT I
260  FOR I=1 TO 10
270  FOR J=1 TO 10
280  LET U(I,J)=V(I,J)
290  NEXT J
300  NEXT I
310  LET V1=0
320  FOR I=2 TO 9
330  FOR J=2 TO 9
340  LET V(I,J)=(V(I+1,J)+V(I-1,J)+V(I,J+1)+V(I,J-1))/4
350  IF ABS((V(I,J)-U(I,J))/V(I,J))<V1 THEN 370
360  LET V1=ABS((V(I,J)-U(I,J))/V(I,J))
370  NEXT J
380  NEXT I
390  LET M=M+1
400  IF V1>.001 THEN 260
410  PRINT "# OF ITERATIONS =";M
420  FOR J=10 TO 1 STEP -1
430  FOR I=1 TO 10
440  PRINT V(I,J),
450  NEXT I
460  PRINT
470  NEXT J
480  END
```

Initialize Potential Matrix

Initialize Interior Values to Average of Boundary Values

Save Old Potentials

Find New Interior Potentials by Averaging Neighbor Values

$\left|\frac{V_{new} - V_{old}}{V_{new}}\right|$

Test for .1% Accuracy

Print out Potential Matrix

The program initializes the potential at interior points of the region to the average of the values on the boundary. It is a property of Laplace's equation that the absolute maximum and absolute minimum values of potential must lie on the boundary. Initializing the interior points to the average value saves some time in the calculation.

It turns out that Laplace's equation has the interesting (and unusual) property that, when you average the nearest neighbor potentials, you can use some old values (from the last iteration over the grid) and some new values (from the present iteration). This allows you to use only one storage matrix for the potential.

Practice Exercise: Modify the program so that you use only the matrix V and average two old and two new nearest neighbor potentials.

A RUN of the program given above looks like:

```
RUN
LAPLAC

# OF ITERATIONS = 34
 9              10             11             12             13
 14             15             16             17             18

 8              9.             9.99882        10.996         11.992
 12.9885        13.9868        14.9885        15.9932        17

 7              8.00119        9.00001        9.99541        10.9885
 11.9815        12.9783        13.9803        14.9885        16

 6              7.00402        8.00461        9.00001        9.99137
 10.9823        11.9767        12.9783        13.9868        15

 5              6.00802        7.01156        8.00865        9.00001
 9.98955        10.9823        11.9815        12.9885        14

 4              5.01156        6.01848        7.01771        8.01047
 9.00001        9.99137        10.9885        11.992         13

 3              4.01325        5.02172        6.0233         7.01771
 8.00865        9.00001        9.99541        10.996         12

 2              3.01155        4.01967        5.02172        6.01848
 7.01156        8.00461        9.00001        9.99882        11

 1              2.00683        3.01155        4.01325        5.01156
 6.00802        7.00401        8.00119        9.             10

 0              1              2              3              4
 5              6              7              8              9
```

Example 2. Find the potential V throughout a square region, given that the potential on the boundary varies linearly from 0 at one corner to 18 at the opposite corner and that a uniform surface charge density of $+.5\epsilon_0$ coulombs per m^2 fills the region.

The problem calls for the solution of Poisson's equation in a situation very similar to that above for Laplace's equation. A program that solves this Poisson equation situation might look like:

```
POISSO

100  DIM V(10,10),U(10,10)
110  READ V(1,1),V(10,1),V(10,10),V(1,10)
120  DATA 0,9,18,9
130  LET V0=V(1,1)+V(1,10)+V(10,10)+V(10,1)
140  FOR I=2 TO 9
150  LET V(1,I)=I-1
160  LET V(10,I)=9+I-1
170  LET V(I,1)=I-1
180  LET V(I,10)=9+I-1
190  LET V0=V0+V(1,I)+V(10,I)+V(I,1)+V(I,10)
200  NEXT I
210  FOR I=2 TO 9
220  FOR J=2 TO 9
230  LET V(I,J)=V0/36
240  NEXT J
250  NEXT I
260  FOR I=1 TO 10
270  FOR J=1 TO 10
280  LET U(I,J)=V(I,J)
290  NEXT J
```

```
300  NEXT I
310  LET V1=0
320  FOR I=2 TO 9
330  FOR J=2 TO 9
340  LET R=.5
350  LET V(I,J)=(V(I+1,J)+V(I-1,J)+V(I,J+1)+V(I,J-1)+R)/4
360  IF ABS((V(I,J)-V(I,J))/V(I,J))<V1 THEN 380
370  LET V1=ABS((V(I,J)-V(I,J))/V(I,J))
380  NEXT J
390  NEXT I
400  LET M=M+1
410  IF V1>.001 THEN 260
420  PRINT "# OF ITERATIONS =";M
430  FOR J=10 TO 1 STEP -1
440  FOR I=1 TO 10
450  PRINT V(I,J),
460  NEXT I
470  PRINT
480  NEXT J
490  END
```

Practice Exercise: Annotate the Poisson program by written comments by each line.

A run of this program looks like:

```
RUN
POISSO

# OF ITERATIONS = 48
 9              10             11             12             13
 14             15             16             17             18

 8              9.58738        10.927         12.1182        13.2048
 14.2044        15.1172        15.9259        16.5866        17

 7              8.92723        10.5112        11.8531        13.0106
 14.0099        14.8513        15.5091        15.9259        16

 6              8.11906        9.85412        11.2951        12.5007
 13.4998        14.2926        14.8513        15.1172        15

 5              7.20652        9.0131         10.5026        11.7322
 12.7311        13.4998        14.0099        14.2044        14

 4              6.2069         8.01384        9.50357        10.7333
 11.7322        12.5007        13.0106        13.2048        13

 3              5.12005        6.85596        8.29763        9.50357
 10.5026        11.2951        11.8531        12.1182        12

 2              3.92835        5.51334        6.85596        8.01384
 9.0131         9.85412        10.5112        10.927         11

 1              2.58811        3.92835        5.12005        6.2069
 7.20652        8.11906        8.92723        9.58738        10

 0              1              2              3              4
 5              6              7              8              9
```

Notice that the final interior values have changed from those of the Laplace solution. The symmetries have been maintained because the charge density is uniform.

Practice Exercise: Identify lines of symmetry in the values of potential.

Terminal plotting can also be used in a manner similar to the terminal plotting of potential done in Chapter One.

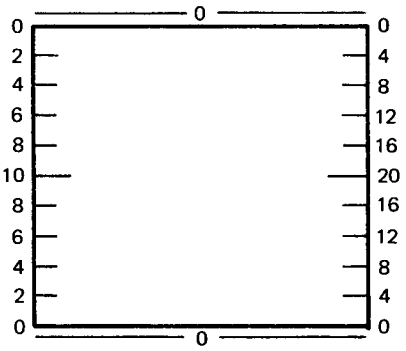
Practice Exercise: Modify the Laplace and Poisson programs to print out characters representing the values of the potential at each point on the grid.

CONCLUSION

The Laplace and Poisson equations are very basic to electrostatics. This chapter showed you one way to solve these equations with the computer. The numerical method of solution presented is also applicable to other partial differential equations arising in physics.

CHAPTER FIVE EXERCISES

1. Consider the square region with the boundary potentials:



(constant zero potential on two sides and increasing linearly to the midpoint then decreasing to zero again on the other sides).

a) Determine the potential inside the region.

b) From a character plot of the potential in the region, draw several equipotentials.

c) Describe the symmetries in the potential.

2. With the boundary potential described in Exercise 1, calculate the potential in the region when there is a uniform charge density of $-.5\epsilon_0$ coulombs/m^2 throughout the region.

a) From a character plot of the potential in the region, draw several equipotentials.

b) Describe the symmetries in the potential.

3. Consider the following line charges parallel to the z-axis: +1 at (6,0); +1 at (–6,0); +1 at (0,6); and +1 at (0,–6).

a) Using the method discussed in Chapter 1, determine the potential in the square defined by the corners [5,5] and [–5,–5]. (Find the value of the potential on a grid with ten points on each side.)

b) Use the values of the potential obtained in Part (a) for the potentials on the edges of the square region defined by the corners [5,5] and [–5,–5]. Use Laplace's equation and determine the potential inside this region.

c) Compare the results obtained by the two methods (Parts (a) and (b)) for interior points.

4. The symmetries observed in the potential pattern in the text and Exercise 1 can be used to cut down the number of points at which the potential needs to be evaluated. For any symmetric pattern, we only need to calculate the potential for a smaller region of space and then use the symmetry to copy the potential into the other regions.

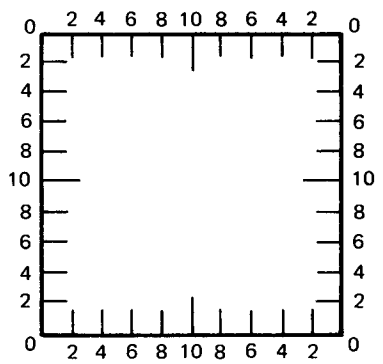
   Consider the potential in Exercise 1. If you consider the pattern centered at the origin the pattern above the x-axis is mirrored below the x-axis. To solve the problem using the symmetry, set the potential the same way for positive y but when you solve for the potential along the x-axis (the line of symmetry), consider the value of each point just below the axis to be the same as the value of the point just above the axis. The grid point equation for this line of symmetry becomes:

$$V_{i,j} = (V_{i+1,j} + V_{i-1,j} + 2V_{i,j-1})/4.$$

   a) Resolve the potential for Exercise 1 taking advantage of the symmetry.

   b) Resolve the potential in Example 2 in the text using the symmetry along the diagonal.

5. Consider the square region with the potential along the edges defined by:



   (with all edges identical, with the corners at zero potentials, and the potential increasing linearly in from the corners to the midpoint where the potential is ten.)

   a) What are the symmetries?

   b) Taking advantage of all of the symmetries, calculate the potential inside the region. Make a character plot of the potential values.

   c) Draw in some of the equipotentials.

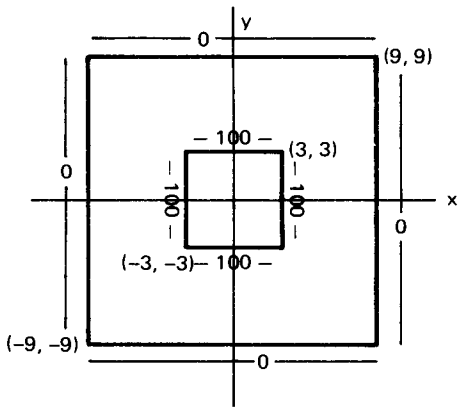
6. Consider the same square region as Exercise 4 but with a uniform charge density of $.5\epsilon_0(|x| + |y|)$ coulombs per m^2 throughout the region.

a) What are the symmetries?

b) Taking advantage of all of the symmetries, calculate the potential inside the region.

c) Make a character plot of the full region and draw the equipotential lines.

7. Consider the following potential: a square boundary with a potential of zero on it. Let the square be defined by the corners (–9,–9) and (9,9). Inside the boundary is a square defined by the corners (–3,–3) and (3,3) at a constant potential of 100. The potential in the region in between can be solved by the techniques described in the text. You must check to see if the point being evaluated is one of the center points with a fixed potential and, if it is, go on to the next point leaving that grid point's potential at 100 as initially defined.

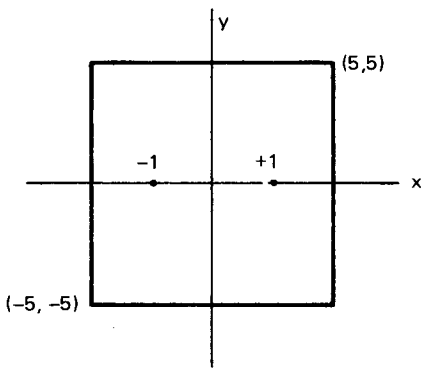


a) What are the symmetries in this problem?

b) Using the symmetries, determine the value of the potential inside the region and make a character plot of the full region.

c) Draw in some of the equipotentials.

8. Consider the potential caused by a +1 and –1 point charge enclosed in a square whose edges are held at zero potential. In order to approximate this situation, assume that the potential near the charge is constant and is unaffected by the rest of the region (i.e., assume the potential near the +1 charge is determined by the equation $k_0 q/r$).



Let the corners of the zero potential box be defined by (–5,–5) and (5,5) and the plus and minus charge positions: +1 at (+2.5,0) and –1 at (–2.5,0).

a) What are the symmetries in the problem?

b) What can you say about the potential on the y-axis?

c) How can the solution of Part (b) be used to simplify this problem?

d) Compute the potential and make an expanded character plot of the potential.

e) Draw in the equipotential lines.

f) Compare your results in shape to the simple dipole charge distribution.

APPENDIX A: TERMINAL PLOTTING

The subroutines in TTYPLO (listed below) allow you to plot a set of curves on your terminal. The subroutines use the letter O variables, so you should not use these variables in your programs. The output from the subroutines is a 5" by 5" x–y plot (assuming 6 lines/inch and 10 characters/inch) with the x-axis across the page and the y-axis up the page.

The range of x values on the plot are set by defining values for X8 = the left side of the plot, X9 = the right side of the plot, Y8 = the bottom of the plot, and Y9 = the top of the plot. The subroutines are called with "GOSUB 9000".

To plot a point on the graph, you define the values of XØ = the x coordinate and YØ = the y coordinate of the point. Also, specify a value for ZØ, which will be the character plotted at the point (XØ,YØ) on the graph. You then call "GOSUB 9100".

The allowed values of ZØ are 0 through 9. Since these variables X8,X9,Y8,Y9,XØ,YØ,ZØ are used in these special ways for the plotting, you should be careful how you use them in your programs.

To produce the final plot on your terminal, call "GOSUB 9200". A sample of the use of TTYPLO follows. The program plots a sine curve (as the character "1").

```
SINE

100  READ X8,X9,Y8,Y9,Z0
110  DATA 0,6.28318,-1,1,1
120  GOSUB 9000
130  FOR X0=X8 TO X9 STEP (X9-X8)/50
140  LET Y0=SIN(X0)
150  GOSUB 9100
160  NEXT X0
170  GOSUB 9200
180  STOP
```

```
APP-TTYPLO
RUN
SINE

MAX Y= 1
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Y            1111
Y          11    11
Y         1        1
Y        1          1
Y
Y       1            1
Y      1              1
Y
Y     1                1
Y    1                  1
Y
Y   1                    1
Y
Y 1                      1
Y
Y1                        1
Y
Y                          1                        1
Y
Y                           1                      1
Y
Y                            1                    1
Y                             1                  1
Y
Y                              1                1
Y                               1              1
Y
Y                                1            1
Y                                 1          1
Y                                  11    11
Y                                    1111
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
MIN Y=-1  MIN X= 0             MAX X= 6.28318

END


TTYPLO

9000  REM INITIALIZE PLOT
9010  DIM O[300]
9020  FOR O1=1 TO 300
9030  LET O[O1]=0
9040  NEXT O1
9050  LET O[52]=52
9060  RETURN
9100  REM STORE POINTS
9105  IF (X0-X8)*(X0-X9)>0 THEN 9165
9110  IF (Y0-Y8)*(Y0-Y9)>0 THEN 9165
9115  IF Z0 >= 0 THEN 9125
9120  LET Z0=0
9125  IF Z0 <= 9 THEN 9135
9130  LET Z0=9
9135  LET O[52]=O[52]+1
9140  LET O[O[52]]=1000*INT(30*(Y0-Y8)/(Y9-Y8)+1.5)
9145  LET O[O[52]]=O[O[52]]+10*INT(50*(X0-X8)/(X9-X8)+1.5)+Z0
9150  IF O[52]<300 THEN 9165
9155  PRINT "PART OF THE PICTURE NOW"
9160  GOTO 9200
9165  RETURN
9200  REM PRINT OUT PLOT
9210  PRINT "MAX Y=";Y9
9220  PRINT " XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX"
9230  FOR O2=31 TO 1 STEP -1
9240  FOR O3=1 TO 51
9250  LET O[O3]=-1
9260  NEXT O3
9270  LET O6=0
9280  FOR O3=53 TO O[52]
9290  IF O[O3]<0 THEN 9360
9300  IF INT(O[O3]/1000) <> O2 THEN 9360
9310  LET O1=O[O3]-1000*INT(O[O3]/1000)
9320  LET O[INT(O1/10)]=O1-10*INT(O1/10)
9330  IF O6>INT(O1/10) THEN 9350
9340  LET O6=INT(O1/10)
9350  LET O[O3]=-1
9360  NEXT O3
9370  PRINT "Y";
9380  FOR O3=1 TO O6
9390  GOTO O[O3]+2 OF 9400,9420,9440,9460,9480,9500,9520,9540,9560,9580,
      9590
```

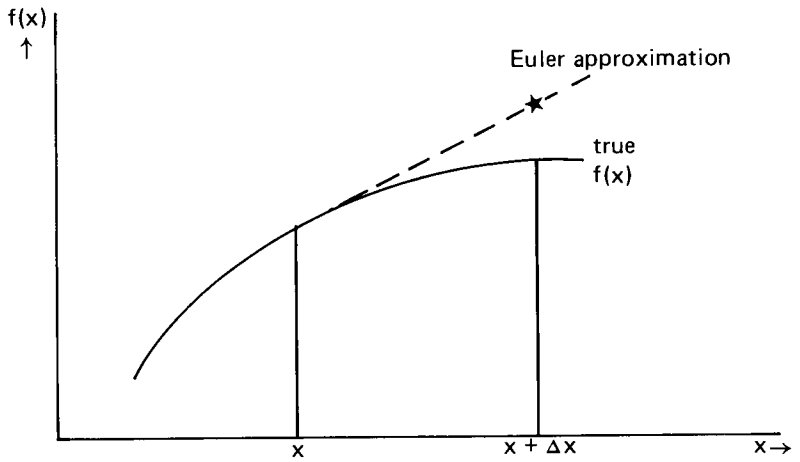
```
9400  PRINT " ";
9410  GOTO 9610
9420  PRINT "0";
9430  GOTO 9610
9440  PRINT "1";
9450  GOTO 9610
9460  PRINT "2";
9470  GOTO 9610
9480  PRINT "3";
9490  GOTO 9610
9500  PRINT "4";
9510  GOTO 9610
9520  PRINT "5";
9530  GOTO 9610
9540  PRINT "6";
9550  GOTO 9610
9560  PRINT "7";
9570  GOTO 9610
9580  PRINT "8";
9590  GOTO 9610
9600  PRINT "9";
9610  NEXT 03
9620  PRINT
9630  NEXT 02
9640  PRINT " XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX"
9650  PRINT "MIN Y=";Y8;"  MIN X=";X8,"MAX X=";X9
9660  GOSUB 9000
9670  RETURN
9680  END
```

APPENDIX B: THE HALF-STEP INTERATIVE INTEGRATION

Numerical methods comprise an entire subject in mathematics. For the moment, all we need to show you is that the half-step method used heavily in the text would seem to give better answers than the simplest approximation (which is called Euler's method). The problem we are discussing is a general one. A physical law often can be stated in terms of the derivative of a function you wish to find; you integrate the differential equation to find the answer.

Consider a general curve for $y=f(x)$; suppose you know the value of y at $x=x_0$ and you have a way to calculate the derivative of y with respect to x anywhere. The problem is to get as good an approximation to the correct value $y=f(x_0+\Delta x)$ as possible. The simplest method (Euler's method) uses the derivative at x_0 and approximates $f(x_0+\Delta x)$ as $f(x_0+\Delta x)=f(x_0)+f'(x_0)\Delta x$.

Euler's method is shown on the sketch by the dashed line. Clearly, the method is correct in the limit as $\Delta x \to 0$. For finite Δx the method produces excessive errors.



The geometrical interpretation of the theorem mathematicians call the *Mean Value Theorem* says that there is some point on the (continuously differentiable) curve $y=f(x)$ such that the derivative at that point has the same value as the slope of the chord from $(x_0,f(x_0))$ to $(x_0+\Delta x,f(x_0+\Delta x))$. If the theorem told us where that point was, everything would be easy. So-called higher order methods of iterative integration (such as the fourth-order Runge-Kutta method or predictor-corrector method) are better and better ways to approximate this "correct" value of slope.

Our half-step method is based on the fact that, in general, evaluating the derivative near the center of the interval $[x_0,x_0+\Delta x]$ is better than using a derivative at the end of the interval. In fact, you can show that, if f(x) is a parabola, then a true half-step method is exact. This means that, if you expand f(x) around x_0 in a Taylor series, the half-step method will be correct through terms in $(\Delta x)^2$. That is why half-step methods are called "second-order" Runge-Kutta methods.

In several cases used in the text of this unit an approximate half-step method, one that approximates $x_0+\Delta x/2$ by using the Δx calculated at the last step, is used instead of a true half-step. Practically always, such an approximate $x_0+\Delta x/2$ lies near the center of the interval $[x_0,x_0+\Delta x]$ and actually increases the accuracy of the method.

Practice Exercise: Using any of the iterative integration programs from the text, compare Euler's method (which uses the derivative at the point x_0) to the half-step method. Show that, for any given step size, the half-step method is more accurate and that, to achieve the same accuracy as the half-step method, a much smaller step size (and hence many more iterations) must be used by Euler's method.

ANSWERS TO SELECTED EXERCISES

CHAPTER ONE

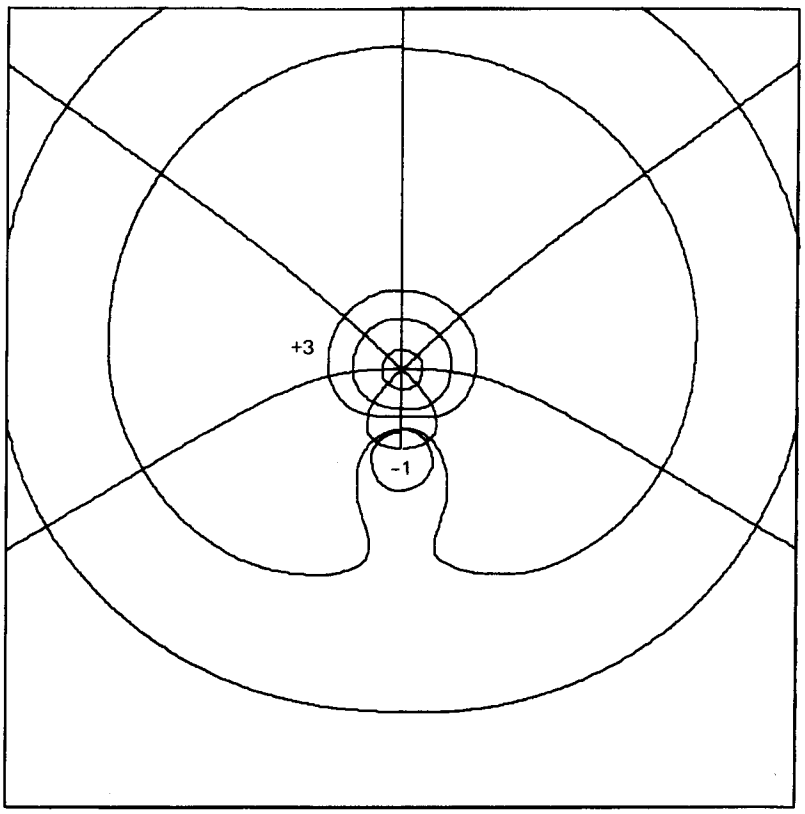
Exercise 1.

```
(X,Y) OF LEFT-BOTTOM & RIGHT-TOP?
? -.55,-.05, -.45,.05
MIN. & MAX. V?
? -100, -11
ZZZYYYYYXXXXXXXXXXXXXXXYYYYYZZZ
ZYYYYXXXXWWWWWWWWWWWWWWXXXXYYYYZ
YYYXXXWWWVVVVVVWWWVVVVVWWXXXYYYY
YYXXXWWVVVUUUTTTTTTTUUUVVWWXXXYY
YXXWWVVVUUTSSRRQQQRRSSTUUVVWWXXY
XXWWVVVUTSRQPOONNNOOPQRSTUVVWWXX
XXWVVVUTSRPOMKJHHHJKMOPRSTUVVWXX
XWWVVUTSRPNLIEA7S7AEILNPRSTUVVWX
XWVVUTSQONIDB6000006DINOQSTUVVWX
XWVVUTRQOLHB1000001BHLOQRTUVVWX
XWVVUTSQONIDB6000006DINOQSTUVVWX
XWWVUTSRPNLIEA7S7AEILNPRSTUVVWX
XXWVVUTSRPOMKJHHHJKMOPRSTUVVWXX
XXWWVVUTSRQPOONNNOOPQRSTUVVWWXX
YXXWWVVUUTSSRRQQQRRSSTUUVVWWXXY
YYXXXWWVVUUUTTTTTTTUUUVVWWXXXYY
YYYXXXWWWVVVVVVWWWVVVVVWWXXXYYYY
ZYYYYXXXXWWWWWWWWWWWWWWXXXXYYYYZ
ZZZYYYYYXXXXXXXXXXXXXXXYYYYYZZZ
```

```
(X,Y) OF LEFT-BOTTOM & RIGHT-TOP?
? .45,-.05, .55,.05
MIN. & MAX. V?
? 40,200
0111222333344444444443333222111 0
1122233444555666666555544332221 1
1223344556777888887776654433221
2234456678899AABBBAA99877665443 22
234456789ABCDEEFEEDCBA987655433
3445679ABDFGIJKLKJIGFDBA9765543
345679ACEHJMPSUUUSPMJGEGA976543
45678ACEHKOTZZZZZZZTOKGECA87654
45679ADFINSZZZZZZZZZSNIFDA97654
45679BDGJOUZZZZZZZZZZUOJGDB97654
45679ADFINSZZZZZZZZZSNIFDA97654
45678ACEHKOTZZZZZZZTOKGECA87654
345679ACEGJMPSUUUSPMJGECA976543
3445679ABDFGIJKLKJIGFDBA9765543
234456789ABCDEEFEEDCBA987655433
2234456678899AABBBAA99876654432 2
1223344556777888887776654433221
1122233444555666666555544332221 1
0111222333344444444443333222111 0
```

```
(X,Y) OF LEFT-BOTTOM & RIGHT-TOP?
? -5,-5,5,5
MIN. & MAX. V?
? 0,5
2222222222223333333333333322222
2222222233333333333333333333322
2222222333333344444443333333332
2222233333334444444444444333333
2222333333344445555555544443333
2222333334445566677666555444333
2223333344455678999887665544433
2223333344456779BEEDCA977655444 3
2233333344444449JTVOHDA97655444 3
22333333444300OSZZZLFB976654443
2233333344444449JTVOHDA97655444 3
2223333344456779BEEDCA977655444 3
2223333344455678999887665544433
2222333334445566677666555444333
2222333333344445555555544443333
2222233333334444444444444333333
2222222333333344444443333333332
2222222233333333333333333333322
2222222222223333333333333322222
```

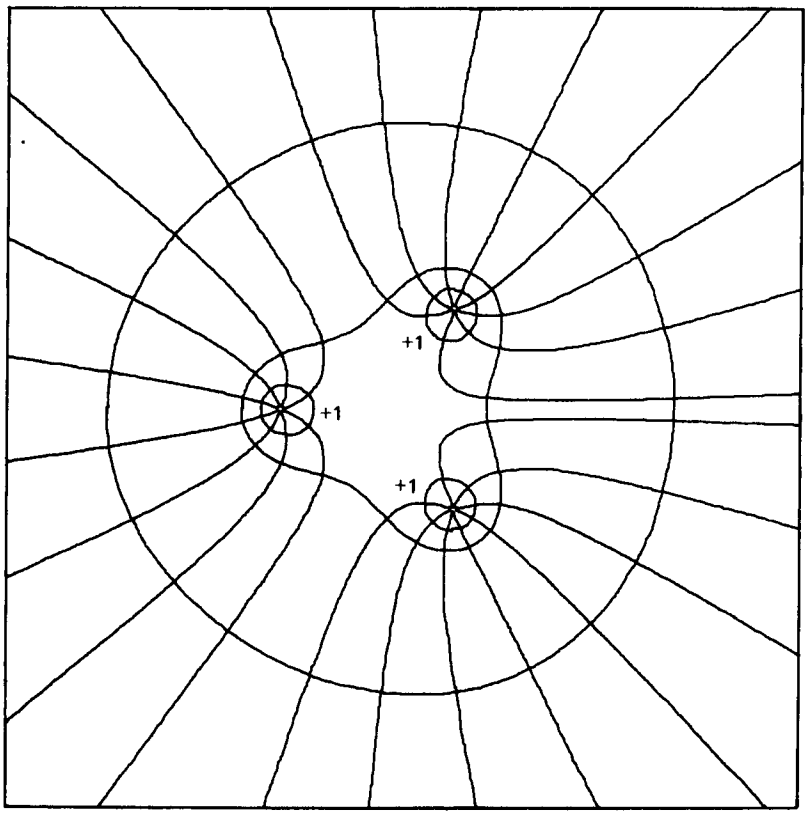
(5, 5)



(-5, -5)

Exercise 2.

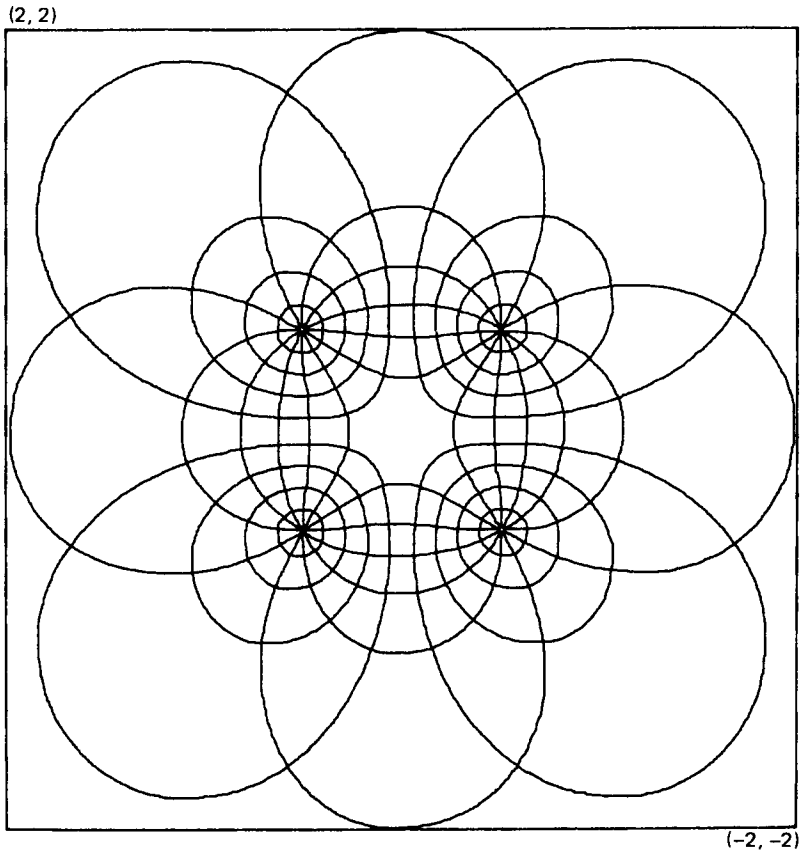
(2, 2)



(-2, -2)

f) Near enough to each line charge, all other charges can be neglected and the equipotential is that of a single line charge (circular). Far away from all three line charges, the equipotential lines are again circular and appear in value as if they were created by a single line charge with a linear charge density of +3.

Exercise 3.

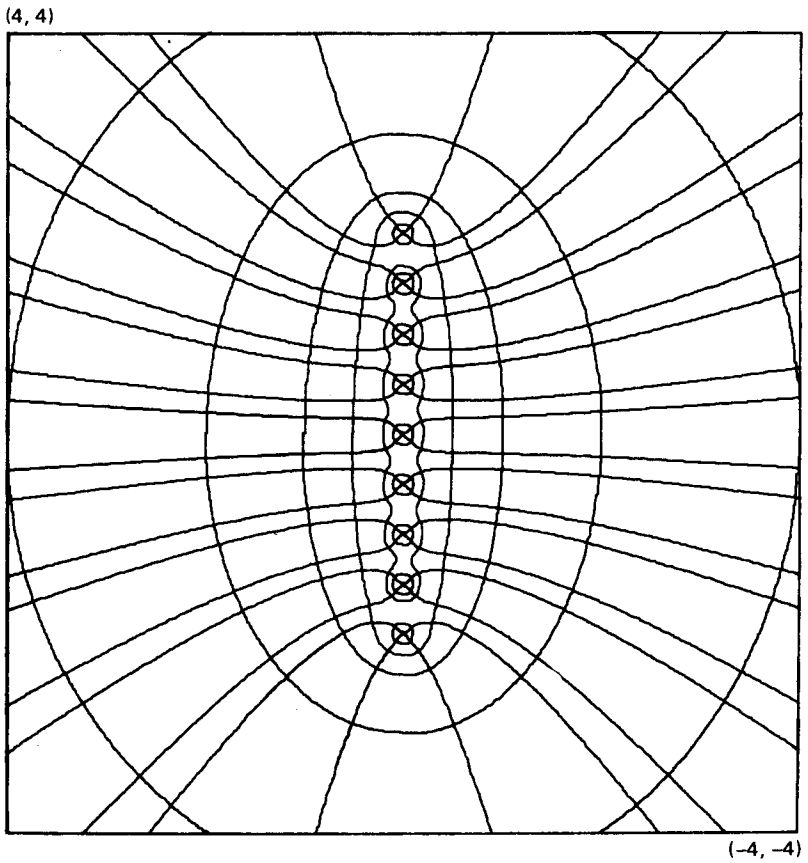


f) The x-z and y-z planes are the zero potential planes.

Exercise 4.

c) Near enough to each line charge, all other charges can be neglected and the equipotential is that of a single line charge (circular). Far away from all three line charges, the equipotential lines are again circular and appear in value as if they were created by a single line charge with a linear charge density of +4.

Exercise 5.



c) The point charge model of a line charge and the true line charge differ in three places: (1) at the ends of the nine charges, (2) very close to the line, and (3) very far away from the line segment.

d) To make a better approximation, increase the number of charge points on the line and make the point charge line segment longer.

Exercise 6.

c) The plane defined by all points with x = .5 and the plane defined by all points with x = –.5 have zero potential.

Exercise 7.

a) A zero potential plane on the y-z plane and the charges +1 at (.5,.5) and –1 at (–.5,+.5).

b) The x-z and y-z planes. The charge is +1 at (.5,.5).

c) Exercise 6 is (an approximation to) the potential for a point charge at the origin and two conducting planes of zero potential defined by the set of prints with x = –.5 and the set of points with x = .5.

d) Solve this by using the following charge distribution: The original charges and the image charges –2 at (–.5,0); +1 at (–.5,.5) and +1 (–.5,–.5).

Exercise 8.

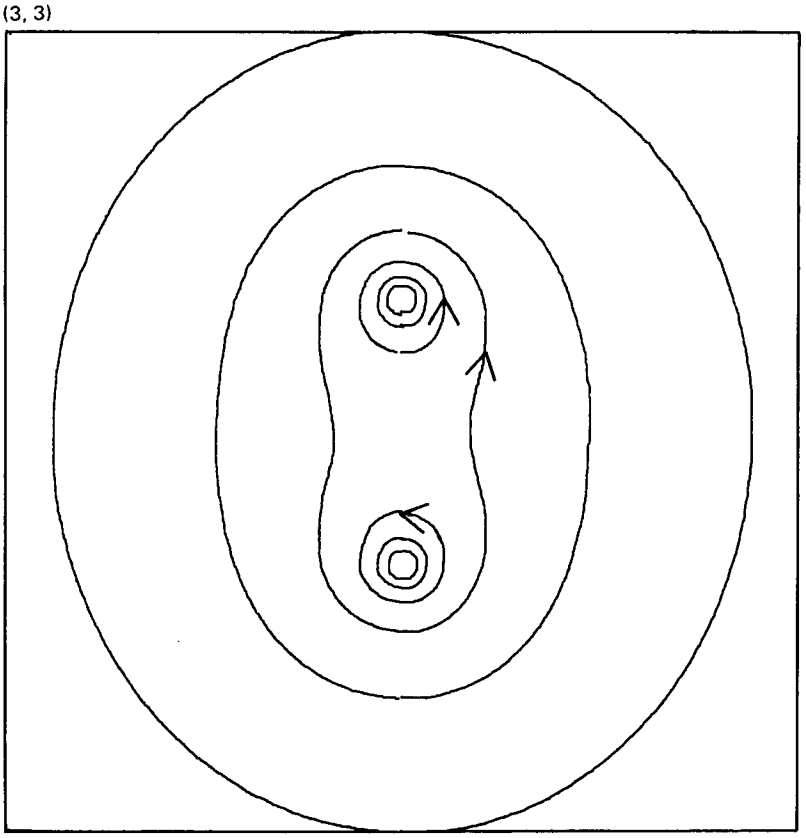
c) The potential inside the cylinder is constant.

d) Gauss's Law is $\oint \vec{E} \cdot \vec{dA} = 4\pi k_0 q$. Since any surface constructed inside the surface will contain no charge, the field is zero, Outside the cylinder, the field is the same as that of a single line charge with a linear charge density equal to the sum of the line charges used to make the cylinder.

CHAPTER TWO

Exercise 1.

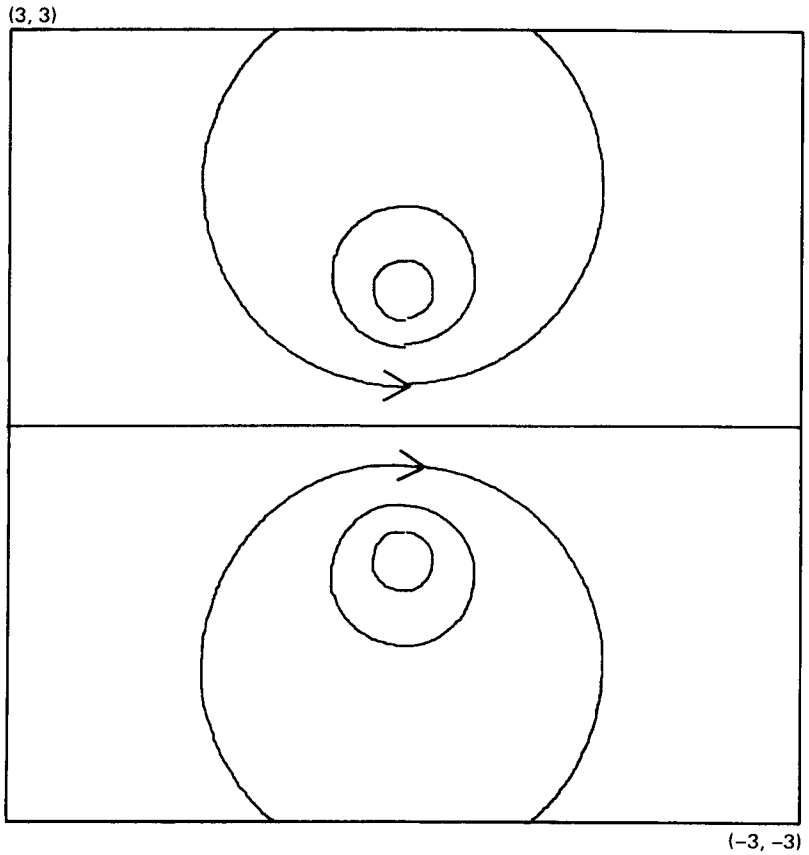
(3, 3)



(–3, –3)

d) Close to each wire, the field is similar to a single current wire of that wire's current. Far away, the field is similar to a single current-carrying wire with a current given by the (algebraic) sum of the currents (for this case +2 current).

Exercise 2.

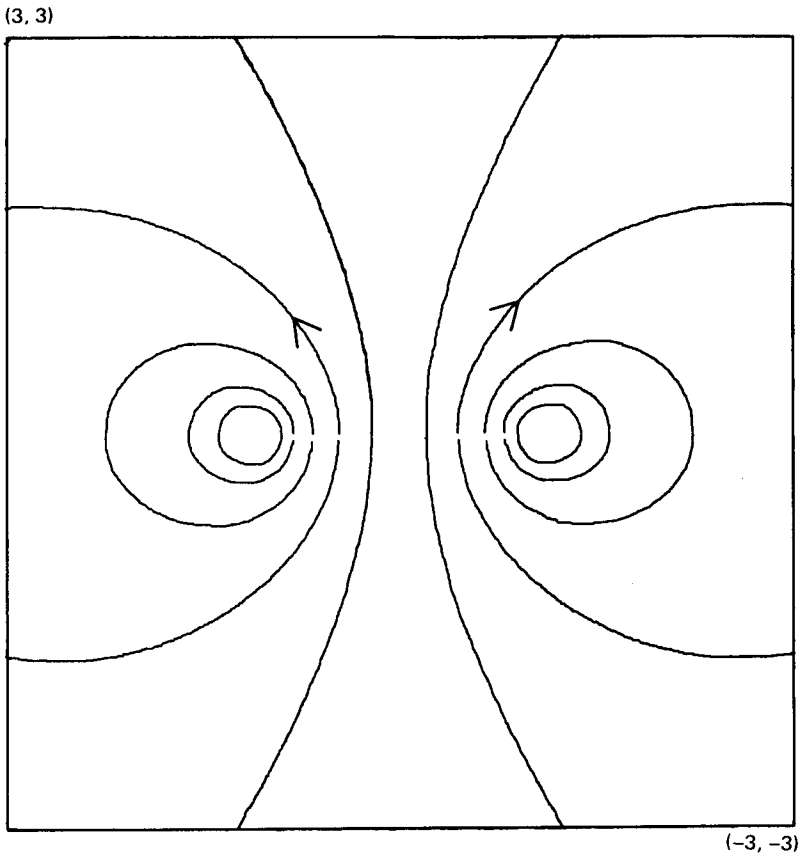


Exercise 3.

b) On the axis of the loop,

$$B_x = \frac{k_0 2\pi I b^2}{(b^2+x^2)^{3/2}}, \quad B_y = B_z = 0$$

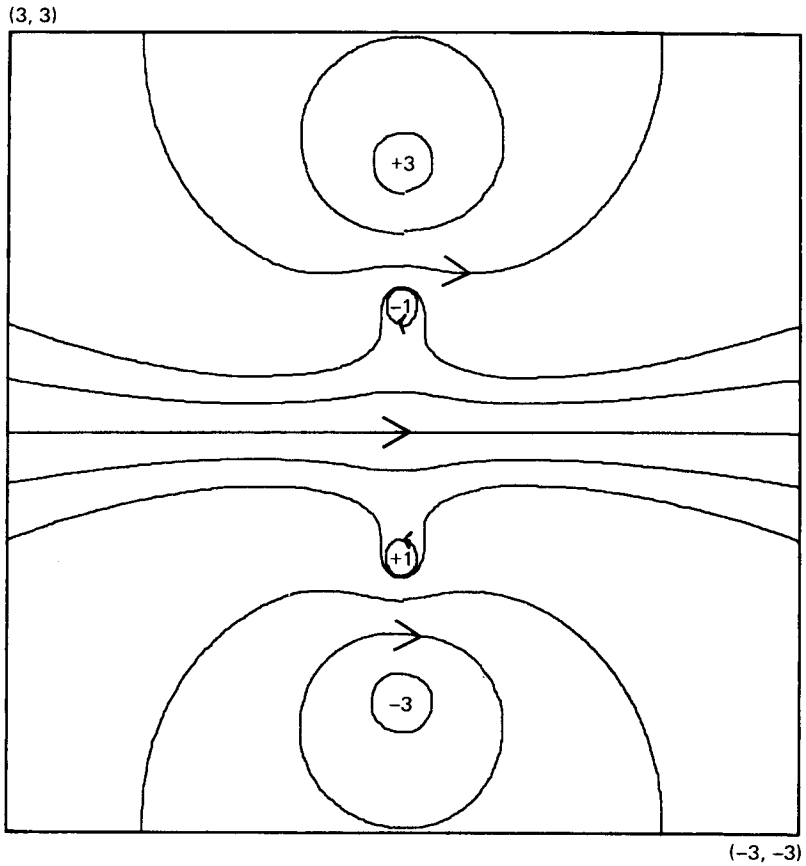
where I is the current, b is the radius of the current loop, and x is the distance along the x-axis. You derive this result by using the Biot-Savart law analytically and noticing that, by symmetry, B_y and B_z are zero. You can integrate the equation for dB_x.



c)

y	Bx
0	6.28319 E-7
0.3	6.74652 E-7
0.6	8.86302 E-7
0.9	2.46628 E-6
1.2	−6.69063 E-7
1.5	−1.78912 E-7
1.8	−8.13398 E-8

Exercise 5.



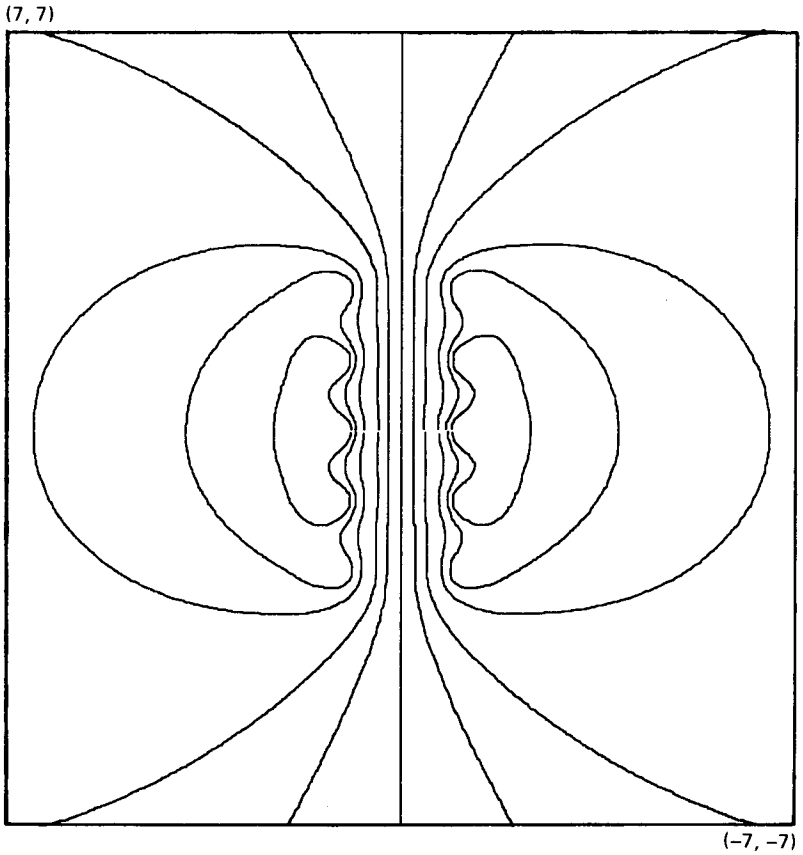
Exercise 7.

a) Inside the field is zero (as well as this approximation will allow). Outside the field is the same as if it were from a single line current of magnitude 16 at the origin.

b) The Biot-Savart law is $\oint \vec{B} \cdot d\vec{\ell} = k_0 I$. Since any line integral inside the cylinder will contain no current, the B field must be zero. Outside the cylinder, any line integral completely outside will contain all the current. Therefore, since the system is radially symmetric it will be the same magnitude and shape as the field from a single conductor at the origin.

c) The field inside both and outside both is zero. The field between the two is $B = 2k_0 I/r$ where r is the radius from the origin. (This agrees with the Biot-Savart law and is similar to the problem of a Toroid.)

Exercise 10.



CHAPTER THREE

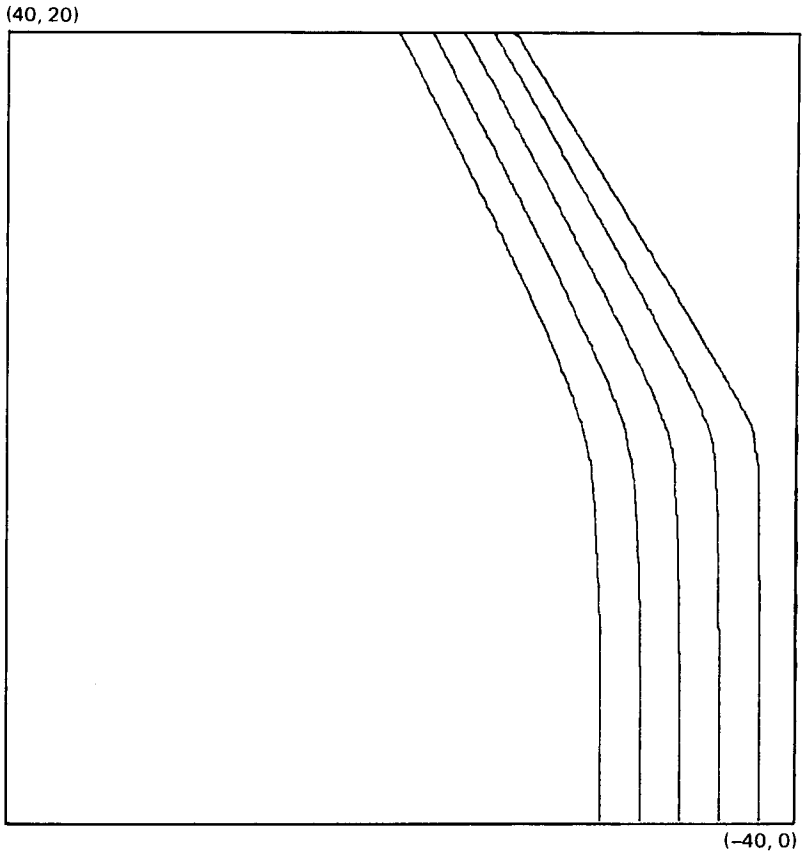
Exercise 1.

a) The trajectory moves away from the line charge but becomes straighter as the positron gets farther away (at constant z velocity).

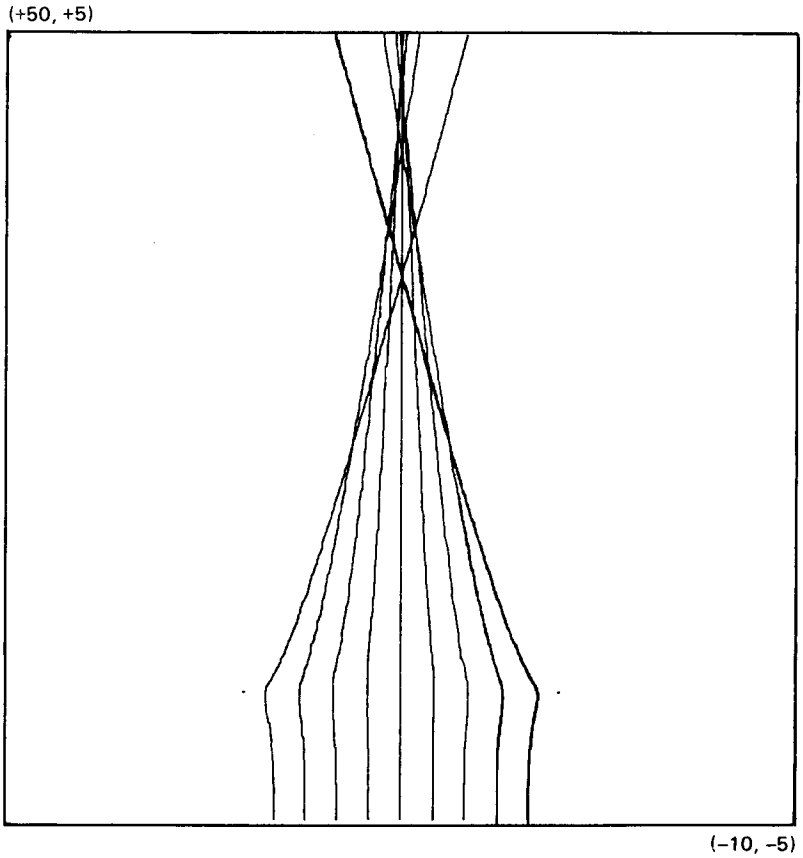
c) Note the angle is almost independent of distance but those really close get bent more.

d) The positron is deflected toward the wire.

e) Yes, but since the force goes as $1/r$ (instead of $1/r^2$), the orbits are (generally) more complicated than elliptical (satellite) orbits.



Exercise 2.



a) The electron will always be equally repelled by both line charges. Therefore, there are never any net forces on the electron and it will remain traveling straight along the z-axis at constant velocity.

b) The electron will always be equally repelled by both line charges. Initially, the electron slows up because of a net force in the –x direction. If the electron did not have energy enough to pass the origin, it would turn around and go back the negative x-axis. Once the electron passes the origin, it will accelerate out the positive x-axis.

c) Yes, and in such a way as to cause the x-axis crossing to be at the same point for all of the trajectories near the center. This can be thought of as a two-dimensional electron lens. You can also see the spherical aberration of trajectories near the line charges.

Exercise 3.

a) The positron circles both charges but the orbit does not close.

b) The positron orbits just the charge at (3,0,0) but is perturbed by the field of the other charge.

c) The positron orbits just the charge at (3,0,0) but is perturbed by the other charge so that its orbit changes.

Exercise 5.

a) The electron's motion is unaffected by the field since the velocity is parallel to the magnetic field.

b) The electron travels with a constant velocity in the z direction but spirals around an axis parallel to the z-axis.

c) The electron just spirals around a circle in the x-y plane.

d) The field collimates the electrons by forcing them to spiral around the magnetic field and not escape in some unwanted direction.

Exercise 6.

a) The positron when traveling into the converging field starts to travel in the y+z direction. Eventually, all of the positron's velocity is gone, and the positron is just traveling in the y-z plane. Then the positron starts coming back out.

b) It decreases to zero, then changes direction.

c) Energy is conserved, V^2 stays constant.

Exercise 7.

b) If the charge is positive, then the positron with $V_x > |E|/|B|$ will move in the +y direction. When $V_x < |E|/|B|$, the positron will move in the −y direction. If the charge is an electron (a negative charge), the results are just reversed.

c) A positively charged particle gets accelerated in the x direction (by $\vec{V}_y \times \vec{B}_z$). This acceleration then causes the velocity, V_x, to grow which, in turn, creates an increase in V_y. If the charge is negative, V_x decreases and V_y increases.

d) The z motion is unaffected. The xy motion is just that already discussed.

e) If there is no change in velocity of the particle, the forces due to E and B must be equal.

$$q\vec{E} = q\vec{V} \times \vec{B}$$

$$Ey = V_x B_z$$

$$V_x = E_y/B_z \qquad \text{with no requirements on } V_z$$

Exercise 9.

$$\vec{F} = m\vec{a}$$

$$\vec{a} = \vec{F}/m$$

$$\vec{V} = \vec{V}_{(initial)} + \vec{a}t$$

$$\vec{S} = \vec{S}_{(initial)} + \vec{V}_{(initial)}\, t + 1/2\, \vec{a}t^2$$

since $\vec{a} = a_y = Eq/m$

$$S_y = S_{y(initial)} + V_{y(initial)}\, t + (1/2)(Eq/m)\, t^2$$

$$S_x = S_{y(initial)} + V_{x(initial)}\, t$$

$$S_z = S_{z(initial)} + V_{z(initial)}\, t$$

For the example in the text:

The initial conditions are $S_x = S_y = S_z = 0$ and initial velocities are $Vy = V_z = 0$ and $V_x = 4$

$$S_y = 1/2\ E\ q/m\ t^2 = 1/2\ t^2$$

$$S_x = 4t$$

$$S_x = 0$$

Exercise 10.

a) $r = \frac{mV}{q|\vec{B}|} = \frac{V}{B}$

when $|\vec{B}|$ is measured in units of q/m.

b) $T = \frac{1}{f} = \frac{2\pi m}{q|\vec{B}|} = \frac{2\pi}{B}$

in normalized units.

CHAPTER FOUR

Exercise 1.

a) The surface integral equals +1 times the length of the box along the z-axis. The line integral is zero.

b) The surface integral equals +2 times the length of the box along the z-axis. The line integral is zero.

c & d) Both the surface and line integrals are zero.

Exercise 2.

a) The surface integral is zero. The line integral is +1 and independent of the contour.

b) The surface integral is zero. The line integral is +2 and independent of the contour.

c & d) Both the surface and line integrals are zero.

Exercise 3.

a) The surface integral is +3 times the length of the box along the z-axis. The line integral is zero.

b) The surface integral is +3 times the length of the box along the z-axis. The line integral is 2.

c & d) Both the surface and line integrals are zero.

Exercise 4.

a) 8. The surface integral is +1 times the total area of the box perpendicular to the z-axis.

b) It is caused by a plane of charge along the x-y axis.

c) zero

Exercise 5.

a) zero

c) 0, 4, 0

d) It is caused by a plane (defined by the y-z plane) of current traveling in the y direction.

Exercise 6.

b) zero

c) $\oint \vec{F} \cdot d\vec{\ell} = \oint \hat{r} f(r) \cdot d\vec{\ell}$

If we pick any circle around the origin, then $\hat{r} \perp d\vec{\ell}$. Therefore the integral is zero.

Exercise 7.

a) zero

b) There are no sources or sinks for the fluid, that is, there are no places where fluid is being created or destroyed.

c) zero

d) –.75

CHAPTER FIVE

Exercise 1.

a)

```
# OF ITERATIONS = 60
 0    0    0    0    0    0    0    0    0    0    0
 2    1.7  1.5  1.4  1.4  1.5  1.7  2.1  2.6  3.3  4
 4    3.4  2.9  2.7  2.6  2.8  3.3  4    5.1  6.4  8
 6    5    4.2  3.7  3.7  3.9  4.6  5.7  7.2  9.4  12
 8    6.3  5.1  4.5  4.3  4.6  5.4  6.8  8.9  11.9 16
 10   7    5.8  4.8  4.6  4.9  5.7  7.2  9.6  13.3 20
 8    6.3  5.1  4.5  4.3  4.6  5.4  6.8  8.9  11.9 16
 6    5    4.2  3.7  3.7  3.9  4.6  5.7  7.2  9.4  12
 4    3.4  2.9  2.7  2.6  2.8  3.3  4    5.1  6.4  8
 2    1.7  1.5  1.4  1.4  1.5  1.7  2.1  2.6  3.3  4
 0    0    0    0    0    0    0    0    0    0    0
```

c) The pattern is symmetric across the x-axis (i.e., across the line joining the +10 and +20 potential points.)

Exercise 2.

a)

```
# OF ITERATIONS = 186
 0    0    0    0    0    0    0    0    0    0    0

 2    1.1  0.5  0.1  0    0.1  0.3  0.8  1.6  2.6  4

 4    2.4  1.3  0.6  0.3  0.4  0.9  1.9  3.4  5.4  8

 6    3.7  2.1  1.1  0.7  0.8  1.6  3    5.1  8.1  12

 8    4.9  2.8  1.5  1    1.2  2.1  3.8  6.5  10.5  16

 10   5.6  3.1  1.7  1.1  1.3  2.3  4.1  7.2  11.9  20

 8    4.9  2.8  1.5  1    1.2  2.1  3.8  6.5  10.5  16

 6    3.7  2.1  1.1  0.7  0.8  1.6  3    5.1  8.1  12

 4    2.4  1.3  0.6  0.3  0.4  0.9  1.9  3.4  5.4  8

 2    1.1  0.5  0.1  0    0.1  0.3  0.8  1.6  2.6  4

 0    0    0    0    0    0    0    0    0    0    0
```

b) The pattern is symmetric across the x-axis (i.e., across the line joining the +10 and +20 potential points.)

Exercise 5.

a) The symmetries are along the x- and y-axes and also along the two diagonals of the square.

b)

```
# OF ITERATIONS = 190
 0

 1    1.8

 2    2.7  3.4

 3    3.5  4.1  4.6

 4    4.4  4.8  5.2  5.6

 5    5.2  5.4  5.7  5.9  6.2

 6    6    6.1  6.2  6.3  6.5  6.6

 7    6.8  6.7  6.6  6.7  6.7  6.8  6.9

 8    7.6  7.2  7    6.9  6.9  6.9  7    7

 9    8.2  7.6  7.3  7.1  7    7    7    7    7.1

 10   8.5  7.8  7.4  7.2  7.1  7    7    7.1  7.1  7.1
```

Exercise 6.

a) The symmetries are along the x- and y-axes and also along the two diagonals of the square.

b)

```
# OF ITERATIONS = 52
 0

 1    1.9

 2    2.8  3.7

 3    3.8  4.8  5.2

 4    4.7  5.3  5.9  6.5

 5    5.5  6.1  6.6  7.1  7.6

 6    6.4  6.8  7.2  7.6  8    8.3

 7    7.2  7.5  7.7  8    8.3  8.6  8.8

 8    8    8    8.2  8.4  8.6  8.8  9    9.2

 9    8.6  8.4  8.8  8.6  8.8  9    9.1  9.3  9.4

 10   9    8.6  8.6  8.7  8.8  9    9.2  9.3  9.4  9.4
```

Exercise 7.

a) The symmetries are along the x- and y-axes and also along the two diagonals of the square.

```
                                                    (9,9)
                                                      0
                                                  2   0
                                              9   5   0
                                         21   14  7   0
                                    37   27   18  9   0
                               60   46   34   22  11  0
                          100  73   55   39   26  13  0
                     100  100  78   59   43   28  14  0
                100  100  100  80   62   45   29  14  0
           100  100  100  100  81   62   46   30  15  0
          (0,0)                                     (9,0)
```

Exercise 8.

a) The pattern is symmetric across the x-axis.

b) The potential along the y-axis is zero.

c) Since we know the potential along the y-axis is zero, we can use the axis as one of the boundaries to the problem. In this way, the region to be solved only contains one charge.