Personal Calculator Algorithms I: Square Roots

A detailed description of the algorithm used in Hewlett-Packard hand-held calculators to compute square roots.

by William E. Egbert

BEGINNING WITH THE HP-35,[1,2] all HP personal calculators have used essentially the same algorithms for computing complex mathematical functions in their BCD (binary-coded decimal) microprocessors. While improvements have been made in newer calculators,[3] the changes have affected primarily special cases and not the fundamental algorithms.

This article is the first of a series that examines these algorithms and their implementation. Each article will present in detail the methods used to implement a common mathematical function. For simplicity, rigorous proofs will not be given, and special cases other than those of particular interest will be omitted.

Although tailored for efficiency within the environment of a special-purpose BCD microprocessor, the basic mathematical equations and the techniques used to transform and implement them are applicable to a wide range of computing problems and devices.

The Square Root Algorithm

This article will discuss the algorithm and methods used to implement the square root function.

The core of the square root algorithm is a simple approximation technique tailored to be efficient using the instruction set of a BCD processor. The technique is as follows:

$\sqrt{x}$ is desired

1. Guess an answer a
2. Generate a^2
3. Find $R = x - a^2$
4. If the magnitude of R is sufficiently small, $a = \sqrt{x}$.
5. If R is a positive number, a is too small.
   If R is a negative number, a is too big.
6. Depending on the result of step 5, modify a and return to step 2.

The magnitude of R will progressively decrease until the desired accuracy is reached.

This procedure is only a rough outline of the actual square root routine used. The first refinement is to avoid having to find a^2 and $x - a^2$ each time a is changed. This is done by finding a one decade at a time. In other words, find the hundreds digit of a, then the tens digit, the units digit, and so on. Once the hundreds digit is found, it is squared and subtracted from x, and the tens digit is found. This process, however, is not exactly straightforward, so some algebra is in order.

The following definitions will be used:

x = the number whose square root is desired
a = most significant digit(s) of $\sqrt{x}$ previously computed
b = the next digit of $\sqrt{x}$ to be found
j = the power of 10 associated with b
$R_a = x - a^2$, the current remainder
a_j = the new a when digit b is added in its proper place. $a_j = a + (b \times 10^j)$ (1)
R_b = the portion of remainder R_a that would be removed by adding b to a. $R_b = a_j^2 - a^2$ (2)

For example, let $x = 54756$. Then $\sqrt{x} = 234$.

Let $a = 200$.
b = the digit we are seeking (3, in this case)
j = 1 (the 10's digit is being computed)
$R_a = 54756 - (200)^2 = 14756$.

Note that a_j and R_b will vary with the choice of b.

The process of finding $\sqrt{x}$ one decade at a time approaches the value of $\sqrt{x}$ from below. That is, at any point in the computation, $a \leq \sqrt{x}$. Consequently, $R_a \geq 0$.

With this in mind it is easy to see that for any decade j, the value of b is the largest possible digit so that

$$R_a - R_b \geq 0$$

or

$$R_b \leq R_a. \quad (3)$$

Using equations 1 and 2 we have

$$R_b = [a + (b \times 10^j)]^2 - a^2.$$

Expanding and simplifying,

$$R_b = 2ab \times 10^j + (b \times 10^j)^2. \quad (4)$$

Inserting (4) into (3) yields the following rule for finding digit b.

Digit b is the largest possible digit so that

$$2ab \times 10^j + (b \times 10^j)^2 \leq R_a \quad (5)$$

When the digit that satisfies equation 5 is found, a new a is formed by adding $b \times 10^j$ to the old a, the decade counter (j) is decremented by 1, and a new R_a is created; the new R_a is the old R_a minus R_b.

Continuing the previous example,

$x = 54756$
$j = 1$
$a = 200$
$x - a^2 = R_a = 14756$

Applying equation 5 to find b:

b	$R_b = 2ab\times10^j+(b\times10^j)^2$	$R_a - R_b$
0	0	14756
1	4100	10656
2	8400	6356
3	12900	1856
4	17600	−2844

Thus $b=3$, since $b=4$ causes overdraft, i.e., $R_a - R_b < 0$. The new $a = 200+3\times10^1 = 230$. The new $R_a = 1856$, the new $j=0$. With these new parameters, the units digit can be found.

This process may seem vaguely familiar, which is not surprising since upon close inspection it turns out to be the (usually forgotten) scheme taught in grade school to find square roots longhand. Of course, trailing zeros and digits are not written in the longhand scheme.

To make this process efficient for a calculator, still another refinement is needed.

$(b\times10^j)^2$ can be expressed as a series, using the fact that the square of an integer b is equal to the sum of the first b odd integers. Thus,

$$(b\times10^j)^2 = b^2\times10^{2j}$$

$$= \sum_{i=1}^{b} (2i-1)\times10^{2j}$$

For example,

$$(3\times10^j)^2 = 1\times10^{2j}+3\times10^{2j}+5\times10^{2j}$$

$$= 9\times10^{2j}$$

Thus $2ab\times10^j+(b\times10^j)^2$ can be expressed as:

$$2ab\times10^j+(b\times10^j)^2 = \sum_{i=1}^{b} 2a\times10^j+(2i-1)\times10^{2j}$$

or

$$R_b = \sum_{i=1}^{b} 2a\times10^j+(2i-1)\times10^{2j} \qquad (6)$$

Now comes a key transformation in the square root routine. It was shown earlier how inequality 3 will give the value b for the next digit of a. Since multiplying both sides of an inequality by a positive constant does not change the inequality, equations 3 and 6 can be multiplied by the number 5.

$$5R_b \leq 5R_a$$

$$5R_b = \sum_{i=1}^{b} 10a\times10^j+(10i-5)\times10^{2j} \qquad (7)$$

b becomes the largest digit so that $5R_b \leq 5R_a$. The new $5R_a$ is equal to the old $5R_a$ minus $5R_b$.

These transformations may seem useless until we examine a few examples of the last term of the right side of (7) for various values of b.

$10a\times10^j+05\times10^{2j}, b=1$
$10a\times10^j+15\times10^{2j}, b=2$
$10a\times10^j+25\times10^{2j}, b=3$

Notice that the two-digit coefficient of 10^{2j} consists of $(b-1)$ and a 5. These two digits will be expressed as $(b-1)|5$ in succeeding equations. 10a is formed by a simple right shift and does not change between terms. If the sum defined in equation 7, as b is incremented by 1, is subtracted from $5R_a$ until overdraft occurs, the digit in the next-to-last digit position is b. Best of all, it is in the exact position to form the next digit of a without further manipulation. Redoing the previous example may help clarify matters.

$R_a = 14756$
$j = 1$
$a = 200$
$5R_a = 73780$

b	$10a\times10^j+(b-1)\|5\times10^{2j}$	$5R_a - 5R_b$
1	20500	53280
2	21500	31780
3	22500	9280 new $5R_a$
4	23500	−14220 overdraft
	new value of a digits	

Notice that when overdraft occurs the new value of a is already created and the new value of $5R_a$ can be found by restoring the previous remainder.

Decrementing the value of j would cause, in effect, $(10a\times10^j)$ to shift right one place, and $(b-1)|5\times10^{2j}$ to shift right two places. The result is that the final 5 shifts one place to the right to make room for a new digit. Continuing with the same example,

$5R_a = 9280$
$a = 230$
$j = 0$

b	$10a \times 10^{i} + (b-1) \mid 5 \times 10^{2j}$	$5R_a - 5R_b$	
1	2305	6975	
2	2315	4660	
3	2325	2335	
4	2335	0	remainder
5	2345	−2345	overdraft
	final $a = \sqrt{x}$		

For ease of understanding, the preceding example treated a large positive number. A number in the calculator actually consists of a mantissa between 1 and 10 and an exponent. The problem is to find the square root of both parts of this argument. Happily, if the input exponent is an even number, the portion of the answer resulting from it turns out to be the exponent of the final answer and is simply the input exponent divided by 2. Thus to find $\sqrt{x}$, the exponent of x is first made even and the mantissa shifted to keep the number the same. The exponent of $\sqrt{x}$ is found by dividing the corrected input exponent by 2. The method described above is then used to find the square root of the shifted input mantissa, which (after possibly being shifted) can be between 1 and 100. The result will then be between 1 and 10, which is the range required for the mantissa of $\sqrt{x}$.

During the process of finding $\sqrt{x}$ the remainder R_a progressively decreases. To avoid losing accuracy, this remainder is multiplied by 10^j after finding each new digit b. This avoids shifting a at all, once the square root extraction process begins. A 12-digit mantissa is generated, which insures accuracy to ±1 in the tenth digit of the mantissa of $\sqrt{x}$.

In summary, the computation of $\sqrt{x}$ proceeds as follows:

1. Generate exponent of answer.
2. Multiply mantissa by 5 to create original $5R_a$
3. With an original a of 0, use the method described above to find 12 b digits to form the mantissa of the answer.
4. Round the mantissa and attach the exponent found previously.
5. Display the answer.

The calculator is now ready for another operation. Σ

References

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