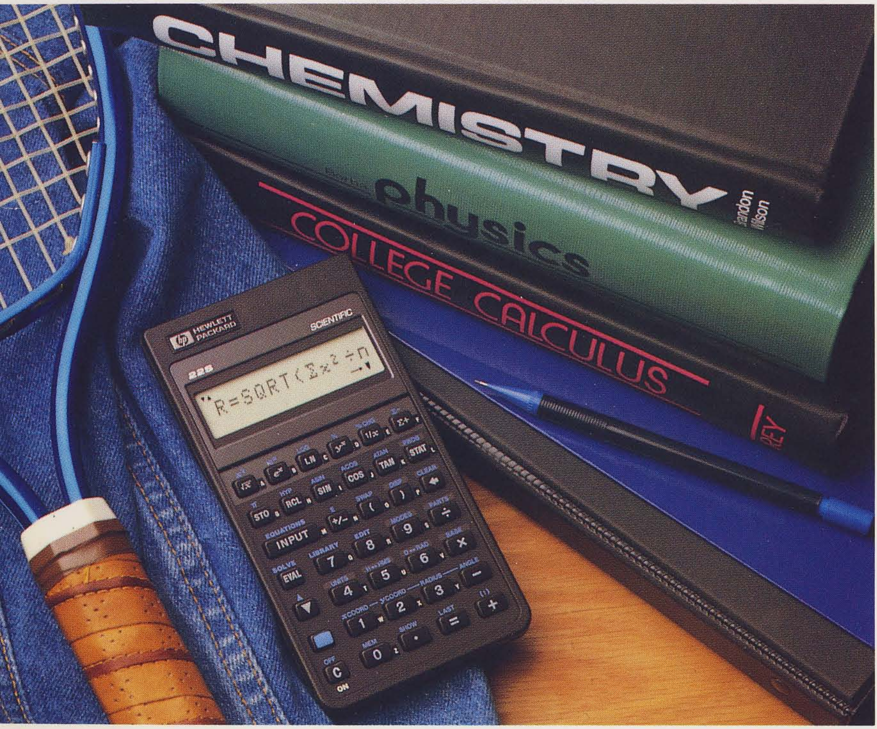
HEWLETT-PACKARD

Step-by-Step Solutions
For Your HP Calculator

Science Student Applications



HP-22S

HEWLETT PACKARD

Science Student Applications

Step-by-Step Solutions for Your HP-22S Calculator

HEWLETT PACKARD

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Contents

How To Use This Book

This book is designed to help you get the most from your HP-22S. It includes:

- Solutions to a number of typical problems you'll encounter in your classwork.
- Several techniques for writing more advanced HP-22S equations.
- A useful compilation of formulas, equations, and information from common areas of study in science and engineering.

Before you try the examples in this book, you should be familiar with certain operations covered in the owner's manual:

- Entering numbers and using the built-in functions (LN, %, TAN, ABS, and so on.)
- Displaying and using menus.
- Storing and recalling variables.
- Storing equations into the list of equations (using ■ [EQUATIONS]).
- Using [EVAL] to evaluate an expression or equation, and using [SOLVE] to solve for an unknown variable in an equation.

Please take a moment to familiarize yourself with the formats used in this book.

Keys and Menu Selection

A box represents a key on the calculator keyboard:

[STAT]
[=]
[STO]
[+]
[INPUT]

The shift key is represented by the symbol ■. Thus, shifted keys appear as:

■ [LAST]
■ [EQUATIONS]
■ [MODES]
■ [SHOW]

A menu key (a key on the top row that has been assigned a new function by a displayed menu) is represented like this:

{GR}	(found in the [MODES] menu)
{Cn,r}	(found in the [PROB] menu)
{J}	(a user-created variable in an equation)

Some menus contain submenus accessed by pressing the appropriate menu key. (These are listed in the owner's manual.) Also, some menus have more than six labels; press {→} to see the other menu options.

Display Formats and Numeric Input

Display Formats. The examples in this book use a display format of four decimal places (FIX 4) except where noted. If your current display format does not match the one used in an example, you can change your display format using the DISP menu (press ■ [DISP]). If you wish to see the full 12-digit precision of a number regardless of the display format, simply press ■ [SHOW]; the full precision number is displayed as long as you hold down the [SHOW] key.

Negative Numbers. Negative numbers are created using the [+/-] key:

Keys:	**Display:**
39.087 [+/-] [=]	-39.0870
2.9 [+/-] ■ [E] 30 [=]	-2.9000E30

Both the [+/-] and [-] keys may be used to create negative exponents of ten:

Keys:	**Display:**
1.408 ■ [E] [+/-] 27 [=]	1.4080E-27
2.55 [+/-] ■ [E] [-] 15 [=]	-2.5500E-15

Clearing Equation Variables. It is not necessary to clear variables before starting an [EVAL] or [SOLVE] operation. However, if you choose to do so, you can clear all variables (A through Z) by pressing ■ [CLEAR] {VAR}. To clear the variables one at a time, store a zero in them, or display the variable of interest in the VARS catalog (■ [MEM] {VARS}) and press ■ [CLEAR]. (For more information, refer to "Clearing Variables" at the end of chapter 2 in your owner's manual.)

Entering Equations into the List of Equations

Equations you enter are stored in the *list of equations*. To enter equations, follow the instructions in the section "Entering Your Own Equations" in chapter 6 of your owner's manual. Here are hints to help you in common error situations:

- The calculator checks the syntax of your equation as you key it in. Thus, it prohibits you from entering equations which make no sense to the calculator. If you press [INPUT] before completing an equation (for example, if the last character is an operator, or if a function is missing an argument), the calculator detects the syntax error and returns the message `INVALID EQN`. When the equation returns to the display, the cursor is at the end of the equation. You can then edit the equation to put it in the correct form. To edit an equation, backspace to the mistake and retype it from there.

- If the calculator accepts the equation but your answer does not match the example, check the values stored in the variables by recalling them (press [RCL], then the proper letter key). If the values are correct, display the equation and check it against the one in this book for accuracy. When you find an error, edit the equation and then press [INPUT].
- If the calculator displays MEMORY FULL when you press [INPUT], you must clear portions of memory before continuing. See "Clearing Portions of Memory" in chapter 1 of the owner's manual for information.

The examples in this book demonstrate approaches to solving certain problems, but by no means exhaust the many possible ways to obtain an answer.

The units used in the examples and elsewhere throughout this book are SI units as described in appendix E.

Our thanks to Steven J. Sabin of Oregon State University for developing the problems and equations in this book.

1

Algebra

This chapter gives examples of problems commonly encountered in algebra:

- Calculating the logarithm with base other than 10 or e.
- Conversions of complex numbers to polar and exponential forms.
- Division of complex numbers.
- Powers of complex numbers.
- Complex roots of a quadratic equation.
- Finding all five roots of a fifth-degree polynomial.

Additional formulas are found in appendix A.

Using Logarithms

The change of base formula states that:

$$\log_b x = \frac{\log_e x}{\log_e b}$$

Example. Find $\log_7 4783$ using the change of base formula:

$$\log_7 4783 = \frac{\log_e 4783}{\log_e 7}$$

Keys:	**Display:**	**Description:**
4783 [LN]	8.4728	Log_e 4783.
[÷] 7 [LN]	÷1.9459	Calculates $\log_e 7$ and
[=]	4.3542	displays the result.

Thus, $\log_7 4783 = 4.3542$.

If you frequently calculate logarithms with bases other than *e* or 10, you may find it easier to use the following HP-22S equation:

L=LN(X)÷LN(B)

Here, L is $\log_b x$, X is x, and B is b. Key this equation into your list of equations, press [INPUT], and follow the next set of keystrokes to repeat the previous example.

Keys:	**Display:**	**Description:**
[EVAL]	X?*value*	Displays current value of X, prompts for the new value.
4783 [INPUT]	X=4,783.0000 B?*value*	Enters 4783 for X, prompts for B.
7 [INPUT]	B=7.0000 L=4.3542	Enters 7 for B, calculates L.

Notice that the HP-22S equation calculates the same result as the previous set of keystrokes.

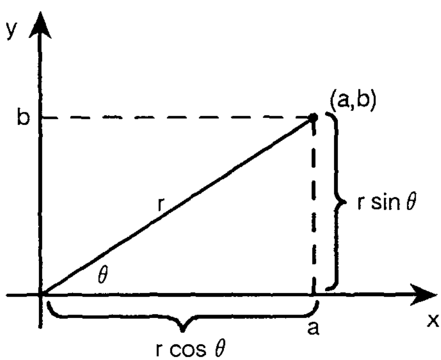
Calculations With Complex Numbers

Polar, Rectangular, and Exponential Forms of Complex Numbers

A complex number z can be expressed in *rectangular* form as

$$z = a + ib$$

where a (the *real part* of z) and b (the *imaginary part* of z) are real numbers and, by definition, $i=\sqrt{-1}$. The number i is often called the *imaginary unit.*



Plotting z in the *complex plane*, where the x-axis corresponds to the real part of z and the y-axis corresponds to the imaginary part of z, generates an alternate form for expressing complex numbers. This is the *polar* form of z:

$$z = r(\cos\theta + i\sin\theta)$$

A third and frequently useful form for complex numbers also exists – the *exponential* form:

$$z = re^{i\theta}$$

where e is the natural logarithm base.

The polar and exponential forms are closely related; r and θ for both forms are computed from the rectangular form using coordinate conversions:

- The Rectangular-to-Polar Conversion

$$r = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\frac{b}{a}$$

- The Polar-to-Rectangular Conversion

$$a = r\cos\theta \qquad b = r\sin\theta$$

The HP-22S provides four keys for converting between rectangular and polar coordinates: ■ [x COORD], ■ [y COORD], ■ [RADIUS], and ■ [ANGLE].

Example. Convert 3 + i4 to polar and exponential forms.

Keys:	Display:	Description:
■ [MODES] {DG}		Sets Degrees mode.
3 ■ [x COORD] 4 ■ [y COORD]	x=3.0000 y=4.0000	Stores real and imaginary parts of rectangular form of the number.
■ [RADIUS] ■ [ANGLE]	r=5.0000 θ=53.1301	Calculates the magnitude and the angle.

Thus, $3 + i4 = 5(\cos 53.1301° + i\sin 53.1301°) = 5e^{i53.1301°}$.

Example. Convert $10e^{i20°}$ to rectangular coordinates:

Keys:	Display:	Description:
■ [MODES] {DG}		Sets Degrees mode.
10 ■ [RADIUS] 20 ■ [ANGLE]	r=10.0000 θ=20.0000	Stores the magnitude and angle of the number.
■ [x COORD] ■ [y COORD]	x=9.3969 y=3.4202	Calculates the real and imaginary parts of the number.

Thus, $10e^{i20°} = 9.3969 + i3.4202$.

Division of Complex Numbers

If the rectangular forms of two complex numbers are converted to their polar forms, complex division can be done using the formula:

$$\frac{z}{w} = \frac{r_z}{r_w} e^{i(\theta_z - \theta_w)}$$

Example. Do the following complex division:

$$\frac{12 + i6.4}{15 - i8.9}$$

Keys:	Display:	Description:
■ [MODES] {DG}		Sets Degrees mode.
12 ■ [x COORD] 6.4 ■ [y COORD]	x=12.0000 y=6.4000	Stores real and imaginary parts of numerator.
■ [RADIUS] [STO] M	r=13.6000	Calculates the magnitude of the numerator and stores it in *M* (for magnitude).
■ [ANGLE] [STO] A	θ=28.0725	Calculates the angle of the numerator and stores it in *A* (for angle).
15 ■ [x COORD] 8.9 [+/-] ■ [y COORD]	x=15.0000 y=−8.9000	Stores real and imaginary parts of denominator.
■ [RADIUS] [STO] [÷] M	r=17.4416	Calculates the magnitude of the denominator, stores magnitude of numerator ÷ magnitude of denominator into *M*.
■ [ANGLE] [STO] [−] A	θ=−30.6821	Calculates the angle of the denominator, stores angle of numerator − angle of denominator into *A*.
[RCL] M ■ [RADIUS]	M=0.7797 r=0.7797	Recalls resulting magnitude. Stores the value in *r*.
[RCL] A ■ [ANGLE]	A=58.7546 θ=58.7546	Recalls resulting angle. Stores the value in θ.
■ [x COORD]	x=0.4045	Calculates real part of division result.
■ [y COORD]	y=0.6666	Calculates imaginary part of division result.

Thus,

$$\frac{12 + i6.4}{15 - i8.9} = 0.40 + i0.67$$

In general, addition and subtraction of complex numbers is easiest when the arguments are in rectangular form. The other operations are best handled when the numbers are in exponential form.

Power of a Complex Number

The following formula can be used to raise a complex number to a real power:

$$(a + ib)^n = \left[re^{i\theta}\right]^n = r^n e^{ni\theta}$$

where

$$r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}\frac{b}{a}$$

Example. Calculate $(5 - i2)^3$.

Keys:	**Display:**	**Description:**
■ [MODES] {DG}		Sets Degrees mode.
5 ■ [x COORD] 2 [+/-] ■ [y COORD]	x=5.0000 y=-2.0000	Stores real and imaginary parts of base.
■ [RADIUS] [y^x] 3 [=]	r=5.3852 156.1698	Calculates the magnitude of the base and cubes it.
■ [RADIUS]	r=156.1698	Stores result as new radius.
[RCL] ■ [ANGLE] [×] 3 [=]	θ=-21.8014 -65.4042	Recalls the angle of the base and multiplies by the exponent (3).
■ [ANGLE]	θ=-65.4042	Stores this new result as a new angle.
■ [x COORD]	x=65.0000	Calculates real part of result.
■ [y COORD]	y=-142.0000	Calculates imaginary part of result.

Thus, $(5 - i2)^3 = 65 - i142$.

Calculating Roots of Polynomials

Complex Roots of a Quadratic Equation

The equation for finding the roots of a quadratic equation $ax^2 + bx + c$ is:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

When $4ac > b^2$, the quantity under the square root is negative. In this case, the roots are complex:

$$x_1 = p + iq \qquad x_2 = p - iq$$

where

$$p = \frac{-b}{2a}$$

and

$$q = \frac{\sqrt{4ac - b^2}}{2a}$$

The following HP-22S equation calculates the complex roots of a quadratic equation. The HP-22S equation uses the control variable* J to determine whether the real or imaginary part of the root is displayed when a root is calculated:

```
R=(J+1)÷2×(-B÷(2×A))+(1-J)÷
  2×(SQRT(4×A×C-SQ(B))÷2÷A
```

Notice that when $J = 1$, the real part of the root $(-b/2a)$ is multiplied by 1 and the imaginary part $(\sqrt{4ac - b^2}/2a)$ is multiplied by 0. When

* Using control variables is discussed in chapter 10 of this book.

$J = -1$, the imaginary part is multiplied by 1 and the real part is multiplied by 0. Thus, depending on the value of J, the real and imaginary parts of the roots are displayed. J must equal 1 or -1 for the equation to produce a meaningful result.

Example. Use the HP-22S equation to calculate the complex roots of $x^2 + 2x + 5$.

Press ■ [EQUATIONS], then [↓] if necessary, to display the message `TYPE NEW EQUATION`. Key the equation into the list of equations and press [INPUT]. Then, use the equation to calculate the complex roots:

Keys:	**Display:**	**Description:**
[EVAL]	J?*value*	Prompts for value of J.
1 [INPUT]	J=1.0000 B?*value*	Stores 1 in J, prompts for B.
2 [INPUT]	B=2.0000 A?*value*	Stores 2 in B, prompts for A.
1 [INPUT]	A=1.0000 C?*value*	Stores 1 in A, prompts for C.
5 [INPUT]	C=5.0000 R=-1.0000	Stores 5 in C, calculates real part of root.
[EVAL]	J?1.0000	Prompts for value of J.
[+/-] [INPUT]	J=-1.0000 B?2.0000	Stores -1 in J, prompts for B.
[INPUT]	B=2.0000 A?1.0000	Stores 2 in B, prompts for A.
[INPUT]	A=1.0000 C?5.0000	Stores 1 in A, prompts for C.
[INPUT]	C=5.0000 R=2.0000	Stores 5 in C, calculates imaginary part of root.

Thus, the two roots of $x^2 + 2x + 5$ are

$$x_1 = -1 + i2$$

and

$$x_2 = -1 - i2$$

Roots of a Fifth-Degree Polynomial

When a polynomial is odd in degree, it always has at least one real root. This root can be found using [SOLVE]. For a fifth-degree polynomial, factoring yields a fourth-degree polynomial. If the roots of this polynomial are complex (and therefore cannot be found using [SOLVE]), you can use the formulas in appendix A to calculate them. However, if there are no x^3 and x terms in the fourth-degree polynomial, the roots can be found using the HP-22S equation for calculating the complex roots of a quadratic equation (page 22).

Example. Find the five roots of $y = x^5 - x^4 - 6x^3 + 6x^2 + 25x - 25$.

Press ■ [EQUATIONS], then [↓] if necessary, to display the message `TYPE NEW EQUATION`. Key in the following equation:

```
X^5-X^4-6×X^3+6×X^2+25×X-25
```

and press [INPUT]. Now, solve for the root:

Keys:	**Display:**	**Description:**
■ [SOLVE]	X=1.0000	Calculates the root.

Use polynomial division to factor out the root:

$$y = (x - 1)(x^4 - 6x^2 + 25)$$

The roots of the resulting fourth-degree polynomial are known to be complex, so you cannot use [SOLVE] to find them. However, notice that if $x^2 = w$, then:

$$x^4 - 6x^2 + 25 = w^2 - 6w + 25$$

The roots of this quadratic in w are a complex conjugate pair and can be found using the HP-22S equation for the complex roots of a quadratic equation (page 22). If you haven't already done so, enter the equation. (If you've already entered the equation, press ■ [EQUATIONS] and use the [↓] key, if necessary, to display it.) Then follow these keystrokes:

Keys:	**Display:**	**Description:**
[EVAL]	J?*value*	Displays current value of J, prompts for new value.
1 [INPUT]	J=1.0000 B?*value*	Stores 1 in J, prompts for B.
6 [+/-] [INPUT]	B=-6.0000 A?*value*	Stores -6 in B, prompts for A.
1 [INPUT]	A=1.0000 C?*value*	Stores 1 in A, prompts for C.
25 [INPUT]	C=25.0000 R=3.0000	Stores 5 in C and calculates the real part of the root.
[EVAL]	J?1.0000	Prompts for value of J.
[+/-] [INPUT]	J=-1.0000 B?-6.0000	Stores -1 in J, prompts for B.
[INPUT]	B=-6.0000 A?1.0000	Stores -6 in B, prompts for A.
[INPUT]	A=1.0000 C?25.0000	Stores 1 in A, prompts for C.
[INPUT]	C=25.0000 R=4.0000	Stores 25 in C and calculates the imaginary part of the root.

Thus, the roots are $w_{1,2} = 3 \pm i\,4$, and:

$$w^2 - 6w + 25 = \left[w - (3 + i4)\right]\left[w - (3 - i4)\right]$$

Since $w = x^2, x = \pm\sqrt{w}$. Therefore:

$$x = \pm\sqrt{3 + i4}$$

and

$$x = \pm\sqrt{3 - i4}.$$

These can be solved using the method shown in the example for calculating the power of a complex number (page 21).

Keys:	**Display:**	**Description:**
■ [MODES] {DG}		Sets Degrees mode.
3 ■ [x COORD] 4 ■ [y COORD]	x=3.0000 y=4.0000	Stores real and imaginary parts of base.
■ [RADIUS] [√x]	r=5.0000 2.2361	Calculates the magnitude of the base and its square root.
■ [RADIUS]	r=2.2361	Stores result in *r* (for radius).
[RCL] ■ [ANGLE] [×] .5 [=]	θ=53.1301 26.5651	Recalls the angle of the base and multiplies by the exponent (.5).
■ [ANGLE]	θ=26.5651	Stores this new result as a new angle.
■ [x COORD] ■ [y COORD]	x=2.0000 y=1.0000	Calculates real and imaginary parts of root 2.

Thus, roots 2 and 3 are ±(2+ i 1). The following keystrokes find the two remaining roots:

Keys:	**Display:**	**Description:**
3 ■ [x COORD] 4 [+/−] ■ [y COORD]	x=3.0000 y= −4.0000	Stores real and imaginary parts of base.
■ [RADIUS] [√x]	r=5.0000 2.2361	Calculates the magnitude of the base and takes the square root (1/2 power).
■ [RADIUS]	r=2.2361	Stores result in *r* (for radius).
[RCL] ■ [ANGLE] [×] .5 [=]	θ= −53.1301 −26.5651	Recalls the angle of the base and multiplies by the exponent (0.5).
■ [ANGLE]	θ= −26.5651	Stores the result as the new θ.
■ [x COORD] ■ [y COORD]	x=2.0000 y= −1.0000	Calculates the real and imaginary parts of root four.

Roots four and five are ± (2 − i 1). Thus, the five roots of the original polynomial are 1, −2 ± i 1, and 2 ± i 1.

2

Trigonometry

This chapter gives examples of common trigonometry problems:

- Velocity of a rotating object.
- Altitude determination.
- Area of a triangle.
- Multiple triangle solution.

Refer to appendix B for additional formulas.

Angular Distance and Speed

The equation for the circumference C of a circle with radius R is:

$$C = 2\pi R$$

and the equation for the average speed of an object that travels distance D in time T is:

$$S = \frac{D}{T}$$

Example. Willy Whippit's sling is 0.5 meters long. Just before the stone is released, it makes one third of a complete revolution in 0.03 seconds. Calculate the approximate speed of the stone as it leaves the sling.

The distance the stone travels in time T = 0.03 seconds is one third the circumference of the circle made by the whirling sling. Thus:

$$S = \frac{D}{T} = \frac{2\pi R/3}{T} = \frac{2\pi(0.5)/3}{(0.03)}$$

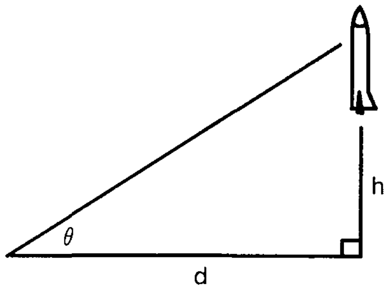
To calculate the answer use the following keystrokes:

Keys:	Display:	Description:
2 [×] ■ [π] [×] .5 [÷] 3 [÷] .03 [=]	34.9066	Calculates S.

The approximate speed of the stone, therefore, is 34.9066 meters per second.

Determining the Altitude of an Object

One way to measure the altitude of an object is to use a device that measures the angle θ between the ground and the object's peak altitude to determine how high it went:



Using the definition of tan θ for a right triangle:

$$\tan\theta = \frac{h}{d} \quad \rightarrow \quad h = d\tan\theta$$

Example. If a man stands 200 feet from the launch pad of a model rocket and measures an angle of 80° when a rocket reaches its maximum altitude, what is the the rocket's maximum altitude?

Keys:	**Display:**	**Description:**
■ [MODES] {DG}		Sets Degrees mode.
200 [×] 80 [TAN] [=]	1,134.2564	Calculates maximum altitude.

The rocket climbs to an altitude of 1,134 feet.

Area of a Triangle

Hero's formula for the area of a triangle with sides A, B, and C is:

$$A = \sqrt{S(S-A)(S-B)(S-C)}$$

where

$$S = \frac{1}{2}(A + B + C)$$

The following HP-22S equations do these calculations:

```
S=.5×(A+B+C)

Q=SQRT(S×(S-A)×(S-B)×(S-C))
```

where Q is the area of the triangle. To enter the equations, press ■ [EQUATIONS], then [↓], if necessary, to display the message `TYPE NEW EQN`. Type each equation and press [INPUT].

Example. A triangular garden plot measures 40, 30, and 50 feet along its three sides. Calculate its area.

Display the first equation. Then:

Keys:	Display:	Description:
[EVAL]	A?*value*	Displays current value of A and prompts for new value.
40 [INPUT]	A=40.0000 B?*value*	Stores 40 in A, prompts for B.
30 [INPUT]	B=30.0000 C?*value*	Stores 30 in B, prompts for C.
50 [INPUT]	C=50.0000 S=60.0000	Stores 50 in C, calculates S.

Press ■ [EQUATIONS] and use the [↓] key to display the second equation (the one that calculates Q). Then:

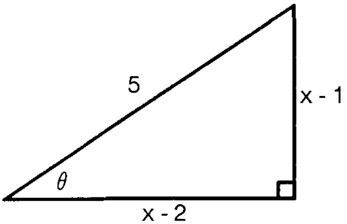
Keys:	Display:	Description:
[EVAL]	S?60.0000	Displays current value of S and prompts for new value.
[INPUT]	S=60.0000 A?40.0000	Stores S, prompts for A.
[INPUT]	A=40.0000 B?30.0000	Stores A, prompts for B.
[INPUT]	B=30.0000 C?50.0000	Stores B, prompts for C.
[INPUT]	C=50.0000 Q=600.0000	Stores C and calculates Q (the area).

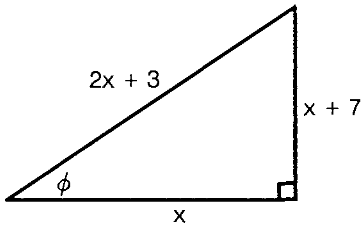
The area is, therefore, 600 square feet.

Multiple Triangles

Sometimes a problem arises that requires certain relationships between two or more triangles to be satisfied. The next example gives such a situation.

Example. For the two right triangles shown below, the angles θ and ϕ must satisfy the equation $\cos(\theta+\phi) = -0.507692$. Find x.





To solve this problem, use the following trigonometric identity:

$$\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$

From the figure above,

$$\cos(\theta) = (x-2)/5$$

$$\cos(\phi) = x/(2x+3)$$

$$\sin(\theta) = (x-1)/5$$

and

$$\sin(\phi) = (x+7)/(2x+3)$$

Substituting into the original equation for $\cos(\theta+\phi)$ gives

$$\cos(\theta+\phi) = \left(\frac{x-2}{5}\right)\left(\frac{x}{2x+3}\right) - \left(\frac{x-1}{5}\right)\left(\frac{x+7}{2x+3}\right) = -0.507692$$

After some simplification, the equation below is obtained. (The equation is simplified so that a shorter HP-22S equation can be derived, resulting in faster solutions.)

$$\frac{x(x-2)-(x-1)(x+7)}{5(2x+3)} = \frac{7-8x}{10x+15} = -0.507692$$

Type the HP-22S equation below into your list of equations.

```
-.507692=(7-8×X)÷(10×X+15)
```

To calculate x, follow the keystrokes below.

Keys:	Display:	Description:
■ [SOLVE]	X=5.0000	Calculates x.

The conditions placed on the two triangles are met when $x = 5$.

3

Linear Algebra

This chapter contains examples from linear algebra:

- Dot product calculation.
- Cross product calculation.
- Finding a perpendicular component of a vector relative to another vector.
- Finding the determinant of a 3×3 matrix.
- Solving a system of simultaneous linear equations using Cramer's Rule.

Additional formulas are found in appendix C.

Calculating a Dot Product

The formula for the dot product of vectors $\mathbf{A} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{B} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$ is:

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The following HP-22S equation calculates dot products.

T=A×D+B×E+C×F

where

$$\begin{array}{ll} A = a_1 & D = b_1 \\ B = a_2 & E = b_2 \\ C = a_3 & F = b_3 \end{array}$$

Example. Calculate $\mathbf{A} \cdot \mathbf{B}$ for the vectors $\mathbf{A} = 3\mathbf{i} + 6\mathbf{j} - 9\mathbf{k}$ and $\mathbf{B} = -1\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$.

Press ■ [EQUATIONS], then [↓], if necessary, to display the message TYPE NEW EQN. Key the equation into the list of equations and press [INPUT]. Then follow the next set of keystrokes.

Keys:	Display:	Description:
[EVAL]	A?*value*	Prompts for A.
3 [INPUT]	A=3.0000 D?*value*	Stores A, prompts for D.
1 [+/-] [INPUT]	D= -1.0000 B?*value*	Stores D, prompts for B.
6 [INPUT]	B=6.0000 E?*value*	Stores B, prompts for E.
2 [INPUT]	E=2.0000 C?*value*	Stores E, prompts for C.
9 [+/-] [INPUT]	C= -9.0000 F?*value*	Stores C, prompts for F.
8 [INPUT]	F=8.0000 T= -63.0000	Stores F, calculates T (dot product).

Thus, $\mathbf{A} \cdot \mathbf{B} = -63$.

Calculating a Cross Product

The formula for the cross product of vectors $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\,\mathbf{i} + (a_3b_1 - a_1b_3)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}$$

The following HP-22S equations calculate the **i**, **j**, and **k** components of the cross product of two vectors:

```
I=B×F-C×E
J=C×D-A×F
K=A×E-B×D
```

where

$$\begin{array}{ll} A = a_1 & D = b_1 \\ B = a_2 & E = b_2 \\ C = a_3 & F = b_3 \end{array}$$

Example. Calculate $\mathbf{A}\times\mathbf{B}$ where $\mathbf{A} = 4\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$ and $\mathbf{B} = -2\mathbf{i} + 3\mathbf{j} + 9\mathbf{k}$.

Enter each equation into the list of equations. When you've entered all three equations, use the arrow keys to display the equation for I. Then follow these keystrokes:

Keys:	Display:	Description:
[EVAL]	B?*value*	Prompts for B.
5 [INPUT]	B=5.0000 F?*value*	Stores B, prompts for F.
9 [INPUT]	F=9.0000 C?*value*	Stores F, prompts for C.
8 [+/-] [INPUT]	C=−8.0000 E?*value*	Stores C, prompts for E.
3 [INPUT]	E=3.0000 I=69.0000	Enters E and calculates I (the **i**-component).

Now, press ■ [EQUATIONS] and use the arrow key to display the equation for J. Then follow these keystrokes:

Keys:	Display:	Description:
[EVAL]	C?−8.0000	Prompts for C.
[INPUT]	C=−8.0000 D?*value*	Stores C, prompts for D.
2 [+/-] [INPUT]	D=−2.0000 A?*value*	Stores D, prompts for A.
4 [INPUT]	A=4.0000 F?9.0000	Stores A, prompts for F.
[INPUT]	F=9.0000 J=−20.0000	Stores F, calculates J (the **j**-component).

Finally, press ■ [EQUATIONS] and use the arrow key to display the equation for K. Then follow theses keystrokes:

Keys:	Display:	Description:
[EVAL]	A?4.0000	Prompts for A.
[INPUT]	A=4.0000 E?3.0000	Stores A, prompts for E.
[INPUT]	E=3.0000 B?5.0000	Stores E, prompts for B.
[INPUT]	B=5.0000 D?-2.0000	Stores B, prompts for D.
[INPUT]	D= -2.0000 K=22.0000	Stores D, calculates K (the **k**-component).

Thus, $\mathbf{A} \times \mathbf{B} = 69\mathbf{i} - 20\mathbf{j} + 22\mathbf{k}$.

Perpendicular Component of a Vector

The component of vector **A** perpendicular to vector **B** is:

$$\mathbf{A} - \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|^2}\mathbf{B}$$

where

$$|\mathbf{B}|^2 = b_1^2 + b_2^2 + b_3^2$$

$\mathbf{A} \cdot \mathbf{B}$, the dot product of the vectors, can be calculated using the HP-22S equation on page 36.

Example. Calculate the component of **A** perpendicular to **B** for the vectors:

$$\mathbf{A} = 4\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$$
$$\mathbf{B} = -2\mathbf{i} + 3\mathbf{j} + 9\mathbf{k}.$$

Calculate $\mathbf{A} \cdot \mathbf{B}$ using the HP-22S equation on page 36. You should obtain a result of `T = -65.0000`. Then, calculate $|\mathbf{B}|^2$:

Keys:	Display:	Description:
[RCL] D ■ [x^2]	4.0000	Calculates b_1^2.
[+] [RCL] E ■ [x^2]	+9.0000	Calculates b_2^2.
[+] [RCL] F ■ [x^2] [=]	94.0000	Calculates $|\mathbf{B}|^2$.

Now, calculate $\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|^2}$:

Keys:	Display:	Description:
[1/x] [×] [RCL] T [=]	-0.6915	Calculates and stores $\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|^2}$.
[STO] K	K= -0.6915	

Next, calculate the three components of $\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|^2}\mathbf{B}$:

Keys:	Display:	Description:
[×] [RCL] D [=]	1.3830	Calculates $\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|^2}b_1$.
[RCL] K [×] [RCL] E [=]	−2.0745	Calculates $\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|^2}b_2$.
[RCL] K [×] [RCL] F [=]	−6.2234	Calculates $\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|^2}b_3$.

Thus:

$$\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|^2}\mathbf{B} = 1.3830\,\mathbf{i} - 2.0745\,\mathbf{j} - 6.2234\,\mathbf{k}$$

To calculate the final result, this vector is subtracted from **A**:

$$\mathbf{A} - \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|^2}\mathbf{B} = (4 - 1.3830)\,\mathbf{i} + (5 - (-2.0745))\,\mathbf{j} + (-8 - (-6.2234))\mathbf{k}$$

Therefore, the component of **A** that is perpendicular to **B** is $2.6170\,\mathbf{i} + 7.0745\,\mathbf{j} - 1.7766\,\mathbf{k}$.

Determinant of a 3 × 3 Matrix

The determinant of a 3 × 3 matrix is given by the equation

$$\det \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = A \begin{vmatrix} E & F \\ H & I \end{vmatrix} - B \begin{vmatrix} D & F \\ G & I \end{vmatrix} + C \begin{vmatrix} D & E \\ G & H \end{vmatrix}$$

The determinant of a 2 × 2 matrix is computed as follows:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The following HP-22S equation calculates the determinant of a 3 × 3 matrix:

```
Q=A×(E×I-F×H)-B×(D×I-F×G)+C×(D×H-E×G)
```

where Q is the determinant, and A through I are the elements of the matrix.

Example. Calculate:

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{vmatrix}$$

Key this equation into the list of equations. Then follow these keystrokes:

Keys:	**Display:**	**Description:**
EVAL	A?*value*	Prompts for A.
1 INPUT	A=1.0000 E?*value*	Stores A, prompts for E.
3 INPUT	E=3.0000 I?*value*	Stores E, prompts for I.
4 INPUT	I=4.0000 F?*value*	Stores I, prompts for F.
3 INPUT	F=3.0000 H?*value*	Stores F, prompts for H.
2 INPUT	H=2.0000 B?*value*	Stores H, prompts for B.
2 INPUT	B=2.0000 D?*value*	Stores B, prompts for D.
1 INPUT	D=1.0000 G?*value*	Stores D, prompts for G.
1 INPUT	G=1.0000 C?*value*	Stores G, prompts for C.
3 INPUT	C=3.0000 Q=1.0000	Stores C, calculates the determinant Q.

Thus,

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 1$$

Simultaneous Equations Solution

A set of three linear equations in three unknowns can be written as:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

where each a_{ij} is a constant coefficient, each c_i is a constant, and each x_j is an unknown.

This system of equations is written compactly in matrix form as

$$\mathbf{Ax} = \mathbf{C}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{(coefficient matrix)}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{(unknown vector)}$$

and

$$\mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad \text{(constant vector)}$$

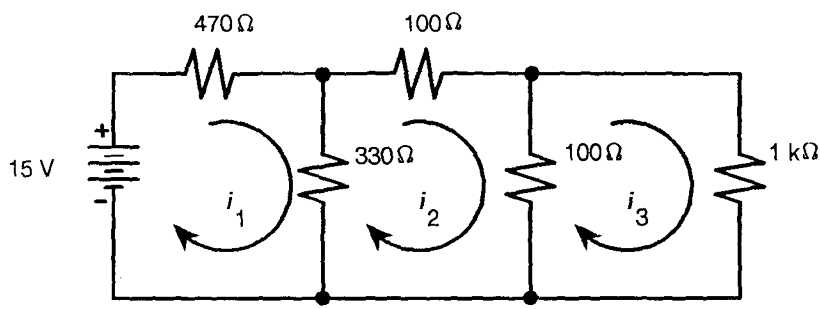
Cramer's Rule states that:

$$x_n = \frac{\det \mathbf{A}_n}{\det \mathbf{A}}$$

where $\mathbf{A}_n$ is the matrix formed by replacing the nth column of **A** with the constant vector **C**.

The HP-22S equation on page 43 can be used to calculate the determinants.

Example. For the circuit below, find i_1, i_2, and i_3.



The following three equations are generated by summing the voltage drops around each loop.

$$\begin{aligned} 800i_1 - 330i_2 + \quad 0i_3 &= 15 \\ -330i_1 + 530i_2 - \quad 100i_3 &= 0 \\ 0i_1 - 100i_2 + 1100i_3 &= 0 \end{aligned}$$

Here,

$$\mathbf{A} = \begin{bmatrix} 800 & -330 & 0 \\ -330 & 530 & -100 \\ 0 & -100 & 1100 \end{bmatrix}$$

and

$$\mathbf{C} = \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix}.$$

Cramer's Rule gives

$$i_1 = \frac{\det \mathbf{A}_1}{\det \mathbf{A}} = \frac{\begin{vmatrix} 15 & -330 & 0 \\ 0 & 530 & -100 \\ 0 & -100 & 1100 \end{vmatrix}}{\begin{vmatrix} 800 & -330 & 0 \\ -330 & 530 & -100 \\ 0 & -100 & 1100 \end{vmatrix}}$$

$$i_2 = \frac{\det \mathbf{A}_2}{\det \mathbf{A}} = \frac{\begin{vmatrix} 800 & 15 & 0 \\ -330 & 0 & -100 \\ 0 & 0 & 1100 \end{vmatrix}}{\begin{vmatrix} 800 & -330 & 0 \\ -330 & 530 & -100 \\ 0 & -100 & 1100 \end{vmatrix}}$$

$$i_3 = \frac{\det \mathbf{A}_3}{\det \mathbf{A}} = \frac{\begin{vmatrix} 800 & -330 & 15 \\ -330 & 530 & 0 \\ 0 & -100 & 0 \end{vmatrix}}{\begin{vmatrix} 800 & -330 & 0 \\ -330 & 530 & -100 \\ 0 & -100 & 1100 \end{vmatrix}}$$

If you haven't already done so, enter the determinant equation (on page 43) into the list of equations. Then, follow the next set of keystrokes.

These keystrokes calculate detA and store the result in variable K:

Keys:	**Display:**	**Description:**
[EVAL]	A?*value*	Prompts for A.
800 [INPUT]	A=800.0000 E?*value*	Stores A, prompts for E
530 [INPUT]	E=530.0000 I?*value*	Stores E, prompts for I.
1100 [INPUT]	I=1,100.0000 F?*value*	Stores I, prompts for F.
100 [+/-] [INPUT]	F=-100.0000 H?*value*	Stores F, prompts for H.
100 [+/-] [INPUT]	H=-100.0000 B?*value*	Stores H, prompts for B.
330 [+/-] [INPUT]	B=-330.0000 D?*value*	Stores B, prompts for D.
330 [+/-] [INPUT]	D=-330.0000 G?*value*	Stores D, prompts for G.
0 [INPUT]	G=0.0000 C?*value*	Stores G, prompts for C.
0 [INPUT]	C=0.0000 Q= 338,610,000.000	Stores C, calculates Q (detA).
[STO] K	K= 338,610,000.000	Stores detA in K for future use.

The next keystrokes calculate det $\mathbf{A}_1$ and i_1:

Keys:	Display:	Description:
[EVAL]	A?800.0000	Prompts for A.
15 [INPUT]	A=15.0000 E?530.0000	Stores A, prompts for E.
[INPUT]	E=530.0000 I?1,100.0000	Stores E, prompts for I.
[INPUT]	I=1,100.0000 F?–100.0000	Stores I, prompts for F.
[INPUT]	F=–100.0000 H?–100.0000	Stores F, prompts for H.
[INPUT]	H=–100.0000 B?–330.0000	Stores H, prompts for B.
[INPUT]	B=–330.0000 D?–330	Stores B, prompts for D.
0 [INPUT]	D=0.0000 G?0.0000	Stores D, prompts for G.
[INPUT]	G=0.0000 C?0.0000	Stores G, prompts for C.
[INPUT]	C=0.0000 Q= 8,595,000.0000	Stores C, calculates Q (det$\mathbf{A}_1$).
[÷] [RCL] K [=]	0.0254	Calculates i_1.

The next keystrokes calculate det A_2 and i_2.

Keys:	Display:	Description:
[EVAL]	A?15.0000	Prompts for A.
800 [INPUT]	A=800.0000 E?530.0000	Stores A, prompts for E.
0 [INPUT]	E=0.0000 I?1,100.0000	Stores E, prompts for I.
[INPUT]	I=1,100.0000 F?-100.0000	Stores I, prompts for F.
[INPUT]	F=-100.0000 H?-100.0000	Stores F, prompts for H
0 [INPUT]	H=0.0000 B?-330.0000	Stores H, prompts for B
15 [INPUT]	B=15.0000 D?0.0000	Stores B, prompts for D.
330 [+/-] [INPUT]	D=-330.0000 G?0.0000	Stores D, prompts for G.
[INPUT]	G=0.0000 C?0.0000	Stores G, prompts for C.
[INPUT]	C=0.0000 Q= 5,445,000.0000	Stores C, calculates Q (detA_2).
[÷] [RCL] K [=]	0.0161	Calculates i_2.

The following keystrokes calculate det A_3 and i_3:

Keys:	**Display:**	**Description:**
[EVAL]	A?800.0000	Prompts for A.
[INPUT]	A=800.0000 E?0.0000	Stores A, prompts for E.
530 [INPUT]	E=530.0000 I?1,100.0000	Stores E, prompts for I.
0 [INPUT]	I=0.0000 F?-100.0000	Stores I, prompts for F.
0 [INPUT]	F=0.0000 H?0.0000	Stores F, prompts for H.
100 [+/-] [INPUT]	H=-100.0000 B?15.0000	Stores H, prompts for B.
330 [+/-] [INPUT]	B=-330.0000 D?-330.0000	Stores B, prompts for D.
[INPUT]	D=-330.0000 G?0.0000	Stores D, prompts for G.
[INPUT]	G=0.0000 C?0.0000	Stores G, prompts for C.
15 [INPUT]	C=15.0000 Q=495,000.0000	Stores C, calculates Q (detA_3).
[÷] [RCL] K [=]	0.0015	Calculates i_3.

Thus, i_1 = 25.4 mA, i_2 = 16.1 mA, and i_3 = 1.5 mA.

4

Calculus

This chapter contains examples from calculus:

- Curve sketching.
- Minimum surface area calculation.
- Numerical integration of a normal probability density.
- Using differentials to approximate changes.
- Taylor series for e.

Additional formulas are found in appendix D.

Using Derivatives

Derivatives give information about how a function changes as its independent variables are changed. Often, this information is used to draw a picture of the function or to find the point at which a function is at a local maximum or minimum. A local maximum or minimum exists at points where the derivative is equal to zero or is undefined.

Example. Sketch the function $y = x^3 + 2x^2 - 5x - 6$. To sketch this function, find the main features of the curve such as its roots (where it crosses the x-axis), where its derivative equals zero (local minimums and maximums), and where its second derivative equals zero (points of inflection).

To find the roots of this function, enter the HP-22S equation

Y=X^3+2×X^2-5×X-6

into your list of equations. Then follow the keystrokes below.

Keys:	Display:	Description:
[INPUT]		Inputs the equation.
0 [STO] X 10	X=0.0000 10_	Stores the boundaries for the root search.
■ [SOLVE]	X Y	Asks which variable to solve for.
{X}	Y?*value*	Displays current value of *Y*, prompts for new value.
0 [INPUT]	Y=0.0000 X=2.0000	Locates a root of 2.

By storing appropriate search boundaries initially, you should repeat the above keystrokes and find roots of −1 and −3 as well. These are the points where the function crosses the *x*-axis. To find the minimums and maximums of the function, compute the derivative and find its roots.

$$\frac{dy}{dx} = 3x^2 + 4x - 5$$

To find the roots of this function, enter

D=3×X^2+4×X−5

into your list of equations. Then follow the keystrokes below.

Keys:	**Display:**	**Description:**
[INPUT]		Inputs the equation.
0 [STO] X 10	X=0.0000 10_	Stores the boundaries for the root search.
■ [SOLVE]	D X	Asks which variable to solve for.
{X}	D?*value*	Displays current value of *D*, prompts for new value.
0 [INPUT]	D=0.0000 X=0.7863	Returns a root of 0.7863.

Storing appropriate search boundaries instructs the equation solver to look for a root in the negative direction (store values of 0 and −100). Repeat the above keystrokes, substituting −100 for 10, and you will find a root at −2.1196.

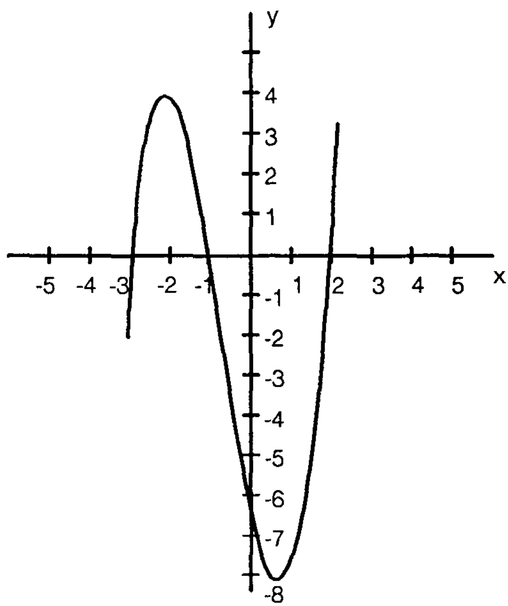
To determine if these values of *x* correspond to a local maximum or minimum, press ■ [EQUATIONS] and use the arrow key to display the equation for *Y*. Evaluate the equation for *Y* at the extremum points.

Keys:	Display:	Description:
[EVAL]	X? −2.1196	Displays the current value of x.
[INPUT]	X= −2.1196 Y=4.0607	The value of y at the extremum.
[EVAL]	X? −2.1196	Displays the current value of x.
0.7863 [INPUT]	X=0.7863 Y= −8.2088	The value of y at the other extremum.

Thus, at $x = -2.1196$ is a local maximum and at $x = 0.7863$ is a local minimum. Now, to make the sketch more accurate, the point of inflection can be found by computing $y''(x)$ and finding its root. This is found as

$$\frac{d^2y}{dx^2} = 6x + 4$$

By inspection, the root of this equation is $x = -4/6$. Thus, the point of inflection (point at which curvature changes from concave up to concave down) is at $x = -0.6667$. The curve is shown below.



Example. The volume of a right circular cone of height h and radius r is given by

$$V = \frac{1}{3}\pi r^2 h$$

and its surface area by

$$A = \pi r\sqrt{r^2 + h^2}$$

A paper drinking cup in the shape of a right circular cone is being designed to hold 250 cubic centimeters of liquid. Find the height h and radius r of a cup which uses the least amount of paper in its construction (has minimum surface area).

From the information given about the volume, h can be expressed in terms of r as

$$V = \frac{1}{3}\pi r^2 h \quad \rightarrow \quad r = \left(\frac{3V}{\pi h}\right)^{\frac{1}{2}} = \left(\frac{750}{\pi h}\right)^{\frac{1}{2}}$$

The expression for the surface area then becomes

$$A = \pi\left(\frac{750}{\pi h}\right)^{\frac{1}{2}}\left(h^2 + \frac{750}{\pi h}\right)^{\frac{1}{2}} = \pi\left(\frac{750h}{\pi} + \frac{750^2}{\pi^2 h^2}\right)^{\frac{1}{2}}$$

To find the minimum surface area, the equation for A is differentiated with respect to h to give

$$\frac{dA}{dh} = \frac{\pi}{2}\left(\frac{750h}{\pi} + \frac{750^2}{\pi^2 h^2}\right)^{-\frac{1}{2}}\left(\frac{750}{\pi} - \frac{(2)(750^2)}{\pi^2 h^3}\right)$$

Key the following equation for $\frac{dA}{dh}$ into your HP-22S list of equations:

```
π÷2×INV(SQRT(750×H÷π+SQ(750)÷SQ(π)
÷SQ(H)))×(750÷π−2×SQ(750)÷SQ(π)÷H^3)
```

Then follow these keystrokes:

Keys:	**Display:**	**Description:**
1 [STO] H	H=1.0000	Avoids a divide by zero error by starting the search for H at a value other than zero.
■ [SOLVE]	H=7.8159	Value of H for which $dA/dh = 0$ (area is minimum at this point).
750 [÷] ■ [π] [÷] [RCL] H [=] [√x]	5.5267	Calculates r from the formula $r = \sqrt{(750/\pi h)}$.

Thus, the necessary dimensions for the paper cup are $r = 5.5267$ cm and $h = 7.8159$ cm.

Numerical Integration

Simpson's Rule is widely used to approximate definite integrals because of its simplicity, accuracy, and ease of implementation.

For integrals of the form

$$\int_a^b f(x)\,dx$$

the approximation for $2n$ * subintervals is given by:

$$S_{2n} = \frac{(b-a)/2n}{3}[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \dots + 2f_{2n-2} + 4f_{2n-1} + f_{2n}]$$

Example. Using Simpson's Rule, approximate

$$y = \frac{1}{\sqrt{2\pi}}\int_0^{.08} e^{-x^2/2}$$

using eight subintervals.

To use Simpson's Rule, divide the interval of integration into the required number of subintervals. Here, the interval length is 0.08 and the number of subintervals is 8. This gives a subinterval length of 0.08/8 or 0.01. The required sum is then

$$S_8 = \frac{0.01}{3}[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + 2f_6 + 4f_7 + f_8]$$

where f_k is $f(0.01k)$.

* The number $2n$ is used because the number of subintervals must be even.

Since the integrand must be evaluated at nine different points, you need to enter the HP-22S equation

```
Y=1÷SQRT(2×π)×EXP(-SQ(X)÷2)
```

in your list of equations. Then follow these keystrokes:

Keys:	**Display:**	**Description:**
[EVAL]	X?*value*	Displays the current value of X.
0 [INPUT]	X=0.0000 Y=0.3989	The value of the integrand at $x = 0$.
[STO] T	T=0.3989	Stores the result in T (for total).
[EVAL]	X?0.0000	Displays the current value of X.
.01 [INPUT]	X=0.0100 Y=0.3989	The value of the integrand at $x = 0.01$.
[×] 4 [=] [STO] [+] T	1.5957	Multiplies by 4 and adds the result to T (for total).

You should continue to do this at $x = 0.02, 0.03, 0.04, 0.05, 0.06, 0.07$, and 0.08. At each step, be sure to multiply the result by the appropriate number (1, 2, or 4) before adding it to T. When you have added the last result to T, press [RCL] T [×] 0.01 [÷] 3 [=] to obtain the value of S_8. You should have a result of 0.0319. It is interesting to note that no known closed-form solution exists for this integral. It represents the area under the standard normal curve in statistics, and it must be approximated numerically.

Using Differentials

If $y = f(x)$ then the differential of y is dy where

$$dy = f'(x)\Delta x$$

Thus, to find a differential dy, simply find $\frac{dy}{dx}$ and multiply by Δx.

The differential dy can be interpreted as the *approximate* change in y due to a change Δx in x. Think of dy as the change that would result in y if it were to change along the fixed line $f'(x)$ as x changes. The true change in y is Δy and is based on y changing along the curve $y = f(x)$, not the fixed line $f'(x)$. For small changes in x, the approximation given by the differential is a good one.

Example. The volume of a conical chemical holding tank is given by

$$V = \frac{1}{12}\pi h^3$$

where h is the height of the liquid above the bottom of the tank. If the chemical level in the tank is at 1 meter and then additional liquid is added so the level raises by 5 cm, approximate the change in volume using differentials and compare this to the exact change.

To find the exact change in volume, compute the volume with $h = 1$ meters and subtract this from the volume calculated with $h = 1.05$ meters. To find the approximate change in volume, find the differential dV as

$$dV = \frac{1}{12}\pi 3(h^2 \Delta h) = \frac{1}{4}\pi(h^2 \Delta h)$$

Here, h is the initial height of the liquid (1 meter), and Δh is the change in liquid height (0.05 meters). The two HP-22S equations

V=1÷12×π×H^3

D=1÷4×π×SQ(H)×C

should be entered into the list of equations in your calculator.

When this is done, use the arrow keys to display the equation for V and follow these keystrokes:

Keys:	**Display:**	**Description:**
[EVAL]	H?*value*	Displays current value of H, prompts for new value.
1 [INPUT]	H=1.0000 V=0.2618	Enters value of H, calculates volume as 0.2618 cubic meters.
[STO] I	I=0.2618	Stores initial volume in I for use later.
[EVAL]	H?1.0000	Displays current value of H, prompts for new value.
1.05 [INPUT]	H=1.0500 V=0.3031	Enters value of H, calculates exact new volume as 0.3031 cubic meters.
[-] [RCL] I [=]	0.0413	Final volume minus initial volume to give true change in volume.

Now, press ■ [EQUATIONS] and use the arrow keys to display the equation for D (differential).

Keys:	**Display:**	**Description:**
[EVAL]	H?1.0500	Displays current value of *H*, prompts for new value.
1 [INPUT]	H=1.0000 C?*value*	Enters value of *H*, prompts for value of *C* (change in *h*).
[INPUT]	C=0.0500 D=0.0393	Inputs change in height (0.05 meters), gives approximate (differential) change in volume.

The true change in volume of 0.0413 cubic meters differs from the differential approximation of 0.0393 cubic meters by only about 5%.

Taylor Series

Many functions of interest can be expressed as an infinite series called a *Taylor Series*. Provided that a function has derivatives of all orders on an interval containing the point a, its Taylor series is given as

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(a) \frac{(x-a)^k}{k!}$$

Example. The Taylor Series expansion for e about the point $a = 0$ is

$$e = \sum_{j=0}^{\infty} \frac{1}{j!}$$

How many terms are needed in the series to express e accurately to four decimal places? The value of e to four decimal places is 2.7183. With the calculator in FIX 4 mode, terms of the sum should be added until this number appears in the display.

Key the HP-22S equation `E=1÷FACT(J)` into your calculator and follow the keystrokes below.

Keys:	Display:	Description:
[EVAL]	J?*value*	Displays current value of J, prompts for new value.
0 [INPUT]	J=0.0000 E=1.0000	Enters value of J, calculates the first term of the sum.
[STO] S	S=1.0000	S is used to save running total of the sum.
[EVAL]	J?0.0000	Displays current value of J, prompts for new value.
1 [INPUT]	J=1.0000 E=1.0000	Increments and enters value of J, calculates the second term of the sum.

[STO] [+] S	E=1.0000	Adds second term of sum to S.
[EVAL]	J?1.0000	Displays current value of J, prompts for new value.
2 [INPUT]	J=2.0000 E=0.5000	Increments and enters value of J, calculates the third term of the sum.
[STO] [+] S	E=0.5000	Adds third term of sum to S.

Continue to press [EVAL], increment J, calculate E, and add the result to S. Be sure to view S ([RCL] S) after each term is added to see when this process can stop. You will find that when $J = 7$ the total in S is 2.7183. Thus, 8 terms ($j = 0$ to 7) are needed to achieve the specified accuracy.

In chapter 10 of this book, this example is repeated with a technique that eliminates the need to do storage register arithmetic in adding each additional term of the series.

5

Physics

This chapter gives examples of solutions to problems frequently encountered in general physics courses:

- Forces and accelerations in a mass and pulley system.
- A simple pendulum.
- The Doppler effect.
- Measuring fluid flow rate with a venturi meter.
- Potential due to an electric dipole.

Force and Acceleration

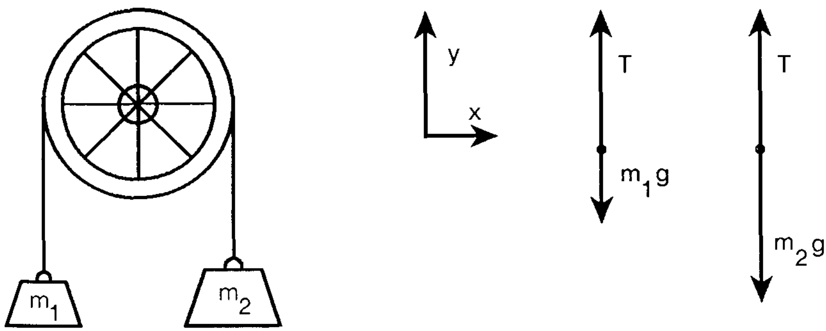
Newton's three laws of motion are the foundation from which a very large class of practical problems in mechanics can be analyzed. The three laws are as follows:

1. A body will continue in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed on it.
2. The sum of forces acting on a body causes it to accelerate in the direction of the resultant of forces. In equation form,

   $$\mathbf{F} = m\mathbf{a}$$

   where **F** is the resultant force acting on the body, m is the mass of the body, and **a** is the acceleration of the body.
3. For every action, there is always an equal and opposite reaction; in other words, if body 1 exerts a force on body 2, body 2 exerts an equal and opposite force on body 1.

Example. The figure below shows a 31.8 kg mass (m_1) and a 55.7 kg mass (m_2) attached by a rope over an ideal pulley (frictionless and massless). Find the acceleration of the masses and the tension in the rope.



The forces acting on each mass are the weight mg (where g is the acceleration due to gravity) and the rope tension T. It is intuitive that the larger mass will accelerate downward and pull the smaller mass upward. Thus, m_2 will accelerate in the negative y-direction. Using Newton's second law for each mass

$$T - m_1 g = m_1 a$$

$$T - m_2 g = -m_2 a$$

Combining these two equations gives

$$a = \frac{m_2 - m_1}{m_1 + m_2} g$$

and

$$T = \frac{2m_1 m_2}{m_1 + m_2} g$$

To calculate T and a, follow the keystrokes below. Recall that g = 9.80665 m/sec^2.

Keys:	**Display:**	**Description:**
9.80665 [÷] [(] 55.7 [+] 31.8 [)] [=] [STO] D	D=0.1121	Stores $\frac{g}{m_1 + m_2}$ in D.
[×] [(] 55.7 [-] 31.8 [)] [=]	2.6786	Calculates a.
[RCL] D [×] 2 [×] 31.8 [×] 55.7 [=]	397.0315	Calculates T.

Thus, the tension in the rope is 397.0315 N and the 55.7 kg mass accelerates downward at 2.6786 m/sec^2.

Simple Harmonic Motion

When the restoring force acting on a body is proportional and opposite to its displacement, the body will oscillate with *simple harmonic motion*. The restoring force equation is written as

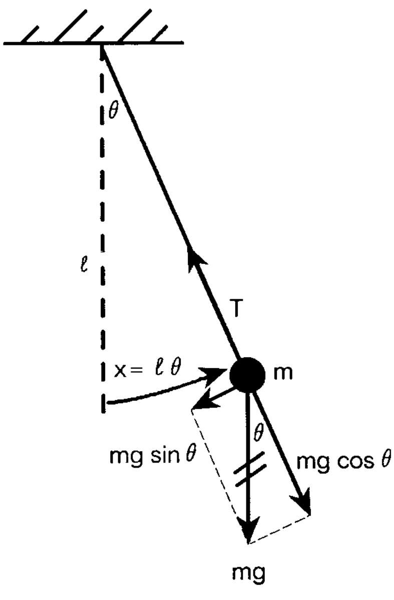
$$F = -k\,x$$

where k is the constant of proportionality. Many physical phenomena can be described as simple harmonic motion. Electrical oscillations in an LC circuit, the movement of a clock pendulum, and the vibration of a guitar string name just a few. The equation of simple harmonic motion of the displacement x is

$$x = A\cos(\omega t + \phi)$$

This is a sinusoid of amplitude A, radian frequency ω and phase constant ϕ. It can be shown that $\omega = \sqrt{k/m}$ where k is the constant of proportionality in the restoring force equation.

Example. The figure below shows a simple pendulum.



As shown in the figure, the restoring force in a simple pendulum is $F = -mg \sin\theta$, and the displacement is given by $x = l\theta$. For small angles $\sin\theta \approx \theta$ and the restoring force becomes

$$F = -mg\theta = -mg\frac{x}{l}$$

Thus, *for small angles* the restoring force is proportional and opposite to the displacement, and the pendulum swings back and forth with simple harmonic motion. Comparing this last equation with the original equation for the restoring force in a simple harmonic oscillator gives

$$F = -kx = -mg\frac{x}{l} \quad \rightarrow \quad k = \frac{mg}{l}$$

Since $T = 2\pi/\omega$, the period of the pendulum can be found as

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}} = \frac{2\pi}{\sqrt{\dfrac{mg}{lm}}} = \frac{2\pi}{\sqrt{g/l}} = 2\pi\sqrt{l/g}$$

Find the period of a simple pendulum with string length $l = 24.8$ cm.

The following HP-22S equation can be used to solve problems involving the simple pendulem.

T=2×π×SQRT(L÷G)

where T is the period, G is the acceleration due to gravity, and L is the string length. Enter it into your list of equations and follow the keystrokes below to solve for T.

Keys:	**Display:**	**Description:**
[EVAL]	L?*value*	Prompts for L.
24.8 ■ [E] [-] 2 [INPUT]	L=0.2480 G?*value*	Stores L, prompts for G.
9.80665 [INPUT]	G=9.8067 T=0.9992	Stores G, calculates T.

Thus, the period of the pendulum is about 1 second.

The Doppler Effect

When a listener and a source of sound are in relative motion towards one another, the sound is heard higher in pitch than if the sources were at rest. In an analogous fashion, the sound is heard lower in pitch if listener and source are receding from one another. This is called the *Doppler effect* and is a familiar phenomena whenever a train goes by; as it approaches the whistle is heard to increase in pitch until the train goes past at which time it decreases in pitch. The equation describing this is

$$f' = f\frac{v \pm v_l}{v \mp v_s}$$

where f' is the frequency heard by the listener, f is the frequency of the sound emitted by the source, v is the speed of sound in the medium, v_l is the speed of the listener relative to the medium, and v_s is the speed of the source relative to the medium. The upper signs (+ in the numerator and − in the denominator) are used when the listener and source move toward each other, and the lower signs are used when the listener and source move away from each other.

Example. The Doppler effect for sound was first tested in 1845 by Buys Ballot. He placed a trumpet player on a railroad flatcar and rolled it toward himself with a constant speed of about 3 m/sec. If the trumpet player blew a note of 440 hz (concert A), calculate the frequency heard by Ballot who stood at the side of the tracks.

The speed of sound in air is about 331.45 m/sec. The listener and source are approaching relative to one another and the listener has a speed of 0 m/sec relative to the medium (air) while the source has a speed of 3 m/sec relative to the medium. The frequency of the source is 440 hz. Thus, the Doppler effect equation becomes

$$f' = f\frac{v \pm v_l}{v \mp v_s} = f\frac{v + 0}{v - 3} = 440\,\frac{331.45 + 0}{331.45 - 3}$$

Keys:	**Display:**	**Description:**
440 [×] 331.45 [÷] [(] 331.45 [-] 3 [)] [=]	444.0189	Calculates $f\,'$.

Thus, the frequency heard by Ballot was about 444 hz.

Fluid Flow

When an incompressible fluid with negligible viscosity flows steadily along a pipe, *Bernoulli's equation* holds at any point along the pipe. It's given by

$$p + \frac{1}{2}\rho v^2 + \rho g y = K$$

where p is the pressure, ρ is the fluid density, v is the fluid velocity, g is the acceleration due to gravity, y is the height of the fluid above some arbitrary reference point, and K is a constant. Thus, at any two points 1 and 2, the equation becomes

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

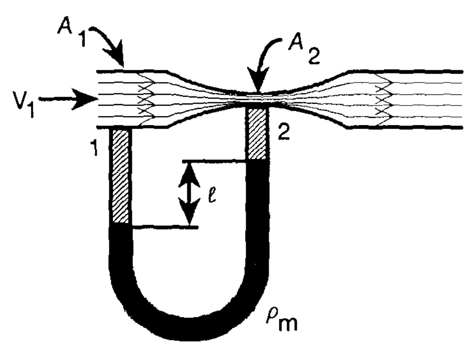
where the fluid is assumed incompressible, and, hence, its density ρ at points 1 and 2 is the same.

The *continuity equation* says that the mass of fluid in a pipe is conserved (none enters or exits the pipe). The continuity equation for an incompressible fluid with steady flow is

$$A_1 v_1 = A_2 v_2$$

where A is the pipe's cross sectional area and v is the fluid's velocity at points 1 and 2, respectively. Informally, the equation states that the flow rate in volume per time is constant at any point in the pipe.

Example. A venturi meter is used to measure speed of fluid flow:



The U-shaped tube in the figure is called a manometer* and is used to measure pressure differences. The manometer equation for pressure difference is

$$p_2 - p_1 = \rho_m g l$$

where p is the pressure at points 1 and 2, ρ_m is the density of fluid in the manometer, g is the acceleration due to gravity, and l is the difference in fluid levels in the manometer. When the continuity equation, Bernoulli's equation, and the manometer equation are combined, the speed of fluid flow at point 1 in the figure is found to be

$$v_1 = A_2\left(\frac{2(\rho_m - \rho)gl}{\rho(A_1^2 - A_2^2)}\right)^{\frac{1}{2}}$$

If the manometer is filled with mercury (density = 13,595 kg/m^3) and the fluid in the pipe is water (density = 1000 kg/m^3), find the velocity of fluid flow at point 1. Assume that the cross sectional area of the pipe at point 1 is 0.073 m^2 and at point 2 is 0.0507 m^2. The difference in mercury levels in the manometer is 9 cm. Follow the keystrokes below to find v_1.

Keys:	**Display:**	**Description:**
[(] 2 [×] [(] 13595 [-] 1000 [)] [×] 9.80665 [×] .09 [)]	22,232.6562	Calculates numerator inside the square root.
[÷] [(] 1000 [×] [(] .073 ■ [x^2] [-] .0507 ■ [x^2] [)] [)] [=]	8,059.6613	Divides numerator by denominator.
[$\sqrt{x}$] [×] .0507 [=]	4.5516	Calculates v_1.

The fluid velocity at point 1 is 4.5516 m/sec.

* The manometer is described in this book in chapter 8 on page 106.

Electric Potential

A charge q will attract other charges of opposite polarity and repel charges of like polarity. The force of attraction or repulsion is given by Coulomb's law, which is described in chapter 7 of your owner's manual. It is convenient to think of a charge as setting up an electric field. Then the force experienced by other charges can be found by determining the strength of the electric field at the point in question.

The equation for the electric potential at a point in an electric field is

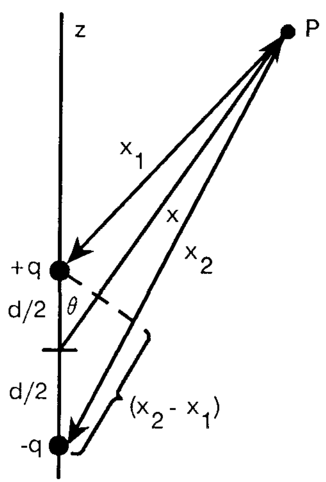
$$V = \frac{W}{q_o}$$

where W is the work required to move a test charge q_o from a point infinitely distant from the field source to the point at which the potential V is desired.

An *electric dipole* is two equal and opposite charges q separated by a distance d. The potential V at a point P is given by the equation below.

$$V = \frac{q}{4\pi\varepsilon_0}\frac{x_2 - x_1}{x_1 x_2}$$

where ε_0 is the permittivity of free space, and the other quantities used in the equation are from the following figure:



If the point P is a large distance away from the dipole, then x is much greater than d ($x >> d$), and the equation for the potential is given approximately as

$$V \approx \frac{qd}{4\pi\varepsilon_0}\frac{\cos\theta}{x^2}$$

Example. An electric dipole consists of two 5 nC charges placed 1.5 meters apart. What is the potential 160 meters away from the center of the dipole at an angle of 30°?

Since $x >> d$ applies in this case, the approximate equation can be used with negligible error. The quantity $1/4\pi\varepsilon_0$ is used frequently in calculations involving electric fields. Its value is 8.9876×10^9 N – m^2/C^2. The HP-22S equation below can be used for dipole calculations.

```
V=8.9876E9×D×Q×COS(A)÷SQ(X)
```

where V is the potential in volts, A is the angle, D is the distance between the charges, Q is the charge of each charge in coulombs, and X is the distance in meters from the center of the dipole. Enter this equation into your list of equations and use the following keystrokes to find V.

Keys:	**Display:**	**Description:**
■ [MODES] {DG}		Sets Degrees mode.
[EVAL]	D?*value*	Prompts for D.
1.5 [INPUT]	D=1.5000 Q?*value*	Stores D, prompts for Q.
5 ■ [E] [-] 9 [INPUT]	Q=5.0000E-9 A?*value*	Stores Q, prompts for A.
30 [INPUT]	A=30.0000 X?*value*	Stores A, prompts for X.
160 [INPUT]	X=160.0000 V=0.0023	Stores X, calculates V.

The potential at this point is about 2.3 millivolts.

6

Chemistry

This chapter gives examples of solutions to problems frequently encountered in general chemisty courses:

- Converting between grams and moles.
- Concentrations of solutions – molarity, molality, and mole fraction.
- Freezing point depression and boiling point elevation of solution.
- pH calculations.
- Faraday's Law.

Converting Between Grams and Moles

One of the most fundamental chemistry calculations is converting between grams and moles. The conversion uses the atomic weight of the element or the molecular weight of the compound.

$$\text{number of moles} = \frac{\text{number of grams}}{\text{atomic or molecular weight}}$$

The following HP-22S equation lets you convert between grams and moles of a substance.

M=G÷W

where M is the number of moles, G is the number of grams, and W is the atomic or molecular weight.

Example. How many moles are in 100 grams of CO_2. (Atomic weights: C=12, O=16.)

Enter the HP-22S equation into the list of equations. Then follow these keystrokes:

Keys:	Display:	Description:
[EVAL]	G?*value*	Prompts for number of grams.
100 [INPUT]	G=100.0000 W?*value*	Stores G, prompts for molecular weight.
12 [+] 16 [×] 2 [INPUT]	W=44.0000 M=2.2727	Stores W, calculates number of moles.

Example. You must weigh out 0.350 moles of $CaCl_2$. How many grams is that? (Atomic weights: Ca = 40.1, Cl = 35.5.)

Keys:	Display:	Description:
■ [SOLVE] {G}	M?*value*	Selects *G*, prompts for number of moles.
.35 [INPUT]	M=0.3500 W?*value*	Stores *M*, prompts for molecular weight.
40.1 [+] 35.5 [×] 2 [INPUT]	 W=111.1000 G=38.8850	Stores *W*, calculates number of grams.

Concentrations of Solutions

Molarity

The *molarity* (M) of a solution is defined as the number of moles of solute per liter of solution. The following HP-22S equation can be used to do molarity calculations:

R=G÷(W×V)

where R is the molarity, G is the number of grams, W is the molecular weight, and V is the volume in liters.

Example. How many grams of KCl are needed to make 250 ml of a .46M solution? (Atomic weights: K=39.1, Cl=35.5)

Enter the HP-22S equation into the list of equations. Then follow these keystrokes:

Keys:	Display:	Description:
■ [SOLVE] {G}	R?*value*	Selects number of grams as the unknown, prompts for molarity.
.46 [INPUT]	R=0.4600 W?*value*	Stores R, prompts for molecular weight.
39.1 [+] 35.5 [INPUT]	W=74.6000 V?*value*	Stores W, prompts for volume.
250 [÷] 1000 [INPUT]	V=0.2500 G=8.5790	Stores V, calculates number of grams.

Example. A solution is prepared by placing 5.25 grams of $MgBr_2$ in a flask and adding water to create 300 ml of solution. Calculate the molarity. (Atomic weights: Mg=24.3, Br=79.9.)

Keys:	**Display:**	**Description:**
[EVAL]	G?*value*	Prompts for number of grams.
5.25 [INPUT]	G=5.2500 W?*value*	Stores *G*, prompts for molecular weight.
24.3 [+] 2 [×] 79.9 [INPUT]	W=184.1000 V?*value*	Calculates and stores *W*, prompts for volume.
300 [÷] 1000 [INPUT]	V=0.3000 R=0.0951	Calculates and stores *V*, calculates molarity.

Molality

The *molality* (*m*) of a solution is the number of moles of solute per 1,000 grams of solvent. The following HP-22S equation can be used to do molality calculations:

L=G×1000÷(W×S)

where *L* is the molality, *G* is the number of grams of solute, *W* is the molecular weight of the solute, and *S* is the number of grams of solvent.

Example. 40.0 g of CCl_4 are added to 150 g of benzene. Calculate the molality of the solution. (Atomic weights: C = 12, Cl = 35.5.)

Keys:	Display:	Description:
[EVAL]	G?*value*	Prompts for grams of solute.
40 [INPUT]	G=40.0000 W?*value*	Stores G, prompts for molecular weight of solute.
12 [+] 35.5 [×] 4 [INPUT]	W=154.0000 S?*value*	Stores W, prompts for grams of solvent.
150 [INPUT]	S=150.0000 L=1.7316	Stores S, calculates the molality.

Example. How many grams of ethanol (molecular weight = 46) must be added to 400 grams of methanol to make a 0.15 m solution.

Keys:	Display:	Description:
■ [SOLVE] {G}	L?*value*	Selects number of grams, prompts for molality.
.15 [INPUT]	L=0.1500 W?*value*	Stores L, prompts for molecular weight of solute.
46 [INPUT]	W=46.0000 S?*value*	Stores W, prompts for grams of solvent.
400 [INPUT]	S=400.0000 G=2.7600	Stores S, calculates grams of solute.

Mole Fraction

The mole fraction of a substance is the ratio of the number of moles of that substance divided by the total number of moles:

$$X_A = \frac{n_A}{n_A + n_B + \cdots}$$

where X_A is the mole fraction of substance A, and n_X is the number of moles of substance X.

Example. You've mixed together 2 g of CO_2, 6 g of N_2, and 5 g of O_2. Calculate the mole fraction of each gas. (Use these molecular weights: CO_2=44, N_2=28, O_2=32.)

Keys:	Display:	Description:
2 [÷] 44 [=] [STO] C [STO] T	0.0455 C=0.0455 T=0.0455	Calculates n_{CO_2}, stores value into *C* and *T*. (*T* will accumulate total number of moles.)
6 [÷] 28 [=] [STO] N [STO] [+] T	0.2143 N=0.2143	Calculates n_{N_2}, stores value into *N*, and accumulates value into *T*.
5 [÷] 32 [=] [STO] [+] T	0.1563	Calculates n_{O_2}, accumulates value into *T*.
[÷] [RCL] T [=]	0.3756	Calculates X_{O_2}.
[RCL] C [÷] [RCL] T [=]	0.1093	Calculates X_{CO_2}.
[RCL] N [÷] [RCL] T [=]	0.5151	Calculates X_{N_2}.

Boiling and Freezing Points of Solutions

When a nonvolatile substance is dissolved in a liquid, the freezing point of the solution is lowered and the boiling point is raised. For dilute solutions, this effect is directly proportional to the molality of the solution:

$$\Delta T_f = m\ K_f \qquad \Delta T_b = m\ K_b$$

where K_f and K_b are constants for a particular solvent.

The following HP-22S equation can be used to do calculations based on either freezing point depression or boiling point elevation. The term *m* (molality) has been replaced by the equation for molality on page 81.

```
T=G×1000×K÷(W×S)
```

where *T* is change in freezing or boiling point, *G* is the number of grams of solute, *K* is the constant, *W* is the molecular weight of the solute, and *S* is the number of grams of solvent.

Example. Calculate the boiling point of a solution prepared by dissolving 25 g of sucrose in 100 g of water. (The molecular weight of sucrose is 342.3; K_b for water is 0.512; the boiling point of water is 100.0 °C.)

Keys:	Display:	Description:
[EVAL]	G?*value*	Prompts for grams of solute.
25 [INPUT]	G=25.0000 K?*value*	Stores *G*, prompts for K_b.
.512 [INPUT]	K=0.5120 W?*value*	Stores *K*, prompts for molecular weight.
342.3 [INPUT]	W=342.3000 S?*value*	Stores *W*, prompts for grams of solvent.
100 [INPUT]	S=100.0000 T=0.3739	Stores *S*, calculates ΔT_b.

[+] 100 [=]	100.3739	Calculates the boiling point.

Example. A solution prepared by dissolving 0.800 g of a substance in 250 g CCl_4 freezes at −23.4 °C. Calculate the molecular weight of the solute. (T_b for CCl_4 is −22.8 °C; K_b for CCl_4 is 29.8.)

Keys:	Display:	Description:
10 [STO] W	W=10.0000	Enters a nonzero guess for *W*. This prevents a possible DIVIDE BY 0 error later.
■ [SOLVE] {W}	T?*value*	Selects molecular weight as the unknown, prompts for freezing point depression.
22.8 [+/-] [-] 23.4 [+/-] [INPUT]	T=0.6000 G?*value*	Stores *T*, prompts for grams of solute.
.8 [INPUT]	G=0.8000 K?*value*	Stores *G*, prompts for K_f.
29.8 [INPUT]	K=29.8000 S?*value*	Stores *K*, prompts for grams of solvent.
250 [INPUT]	S=250.0000 W=158.9333	Stores *S*, calculates molecular weight of solute.

pH Calculations

The pH is a measure of the acidity of a solution. It is defined by the equation:

$$pH = -\log[H^+]$$

where $[H^+]$ is the hydrogen ion concentration. If the pH is known, $[H^+]$ can be calculated by the equation:

$$[H^+] = 10^{-pH}$$

Example. Calculate the pH of the following solutions:

$$[H^+] = 2.5\times10^{-5}$$
$$[H^+] = 6.9\times10^{-6}$$
$$[H^+] = 9.2\times10^{-10}$$

Keys:	Display:	Description:
2.5 ■ [E] [-] 5 ■ [LOG] [+/-]	4.6021	Calculates pH of first solution.
6.9 ■ [E] [-] 6 ■ [LOG] [+/-]	5.1612	Calculates pH of second solution.
9.2 ■ [E] [-] 10 ■ [LOG] [+/-]	9.0362	Calculates pH of third solution.

Example. Calculate [H^+] for solutions with pH's of 3.1 and 9.8.

Keys:	Display:	Description:
■ [DISP] {SC} 4		Sets display to scientific notation.
3.1 [+/-] ■ [10^x]	7.9433E-4	Calculates [H^+] of first solution.
9.8 [+/-] ■ [10^x]	1.5849E-10	Calculates [H^+] of second solution.
■ [DISP] {FX} 4	1.5849E-10	Restores FIX 4 display format.

Ionization of Water

Water ionizes according to the chemical equation:

$$H_2O \rightarrow H^+ + OH^-$$

The following equation for the ionization constant of water (K_w) describes the relationship between the concentrations of H^+ and OH^-.

$$K_w = [H^+][OH^-] = 1\times10^{-14}$$

The equation for K_w can be expressed as the HP-22S equation:

```
H×O=1E-14
```

This relationship lets you calculate $[H^+]$ when $[OH^+]$ is known, or vice versa.

Example. Calculate the pH of 0.18M NaOH.

Enter the HP-22S equation into the list of equations. Then follow these keystrokes:

Keys:	**Display:**	**Description:**
■ [SOLVE] {H}	O?*value*	Selects *H*, prompts for *O*.
.18 [INPUT]	O=0.1800 H=5.5556E-14	Calculates $[H^+]$.
■ [LOG] [+/-]	13.2553	Calculates the pH.

Example. What molarity NaOH solution has a pH of 11.2?

Keys:	Display:	Description:
■ [SOLVE] {O}	H?*value*	Selects O, prompts for H.
11.2 [+/-] ■ [10^x] [INPUT]	H=6.3096E-12 O=0.0016	Calculates $[OH^-]$, the molarity of NaOH.

Faraday's Law

Faraday's Law states that the amount of a substance liberated at an electrode is directly proportional to the amount of current passing through the electrode. A *faraday* is defined as 96,487 coulombs and is the quantity of charge of one mole of electrons.

The following HP-22S equation uses the value of the faraday to define a relationship between current and the amount of a substance liberated or consumed by electrolysis.

```
G=T×I×W÷(96487×N)
```

where G is the number of grams of the substance, T is the time (in seconds), I is the current (in amperes), W is the atomic weight, and N is the number of moles of electrons transferred per mole of substance. For example, $N=2$ for the reaction $Cu^{+2} + 2e^{-} \rightarrow Cu$; likewise, $N=1$ for reaction $K \rightarrow K^{+} + e^{-}$.

Example. How many grams of copper are plated out onto an electrode during the electrolysis of $CuSO_4$ if a current of 0.50 amp flows for 15 minutes? (The atomic weight of Cu is 63.5.)

Enter the HP-22S equation into the list of equations. Then follow these keystrokes:

Keys:	**Display:**	**Description:**
[EVAL]	T?*value*	Prompts for number of seconds.
15 [×] 60 [INPUT]	T=900.0000 I?*value*	Stores T, prompts for current.
.5 [INPUT]	I=0.5000 W?*value*	Stores I, prompts for atomic weight.
63.5 [INPUT]	W=63.5000 N?*value*	Stores W, prompts for number of electrons.

2 [INPUT]	N=2.0000 G=0.1481	Stores N, calculates number of grams of Cu.

Example. How long does it take to plate 5 g of Ag onto an electode during the electrolysis of $AgNO_3$ using a current of 0.8 amps. (The atomic weight of Ag is 107.9.)

Keys:	Display:	Description:
■ [SOLVE] {T}	G?*value*	Selects T, prompts for grams.
5 [INPUT]	G=5.0000 I?*value*	Stores G, prompts for current.
.8 [INPUT]	I=0.8000 W?*value*	Stores I, prompts for atomic weight.
107.9 [INPUT]	W=107.9000 N?*value*	Stores W, prompts for number of electrons.
1 [INPUT]	N=1.0000 T=5,588.9133	Stores N, calculates number of seconds.
[÷] [(] 60 [×] 60 [)] [=]	1.5525	Calculates number of hours.
■ [H↔HMS] {→HMS}	1.3309	Converts to hours-minutes-seconds (1 hour, 33 minutes, 9 seconds).

7

Statics and Dynamics

This chapter contains examples from both statics and dynamics:

- Equilibrium of cable tensions.
- Supporting force of a gear reducer.
- Resultant force on a beam.
- Velocity of a plane being tracked by radar.
- Velocity of a crate sliding with friction.

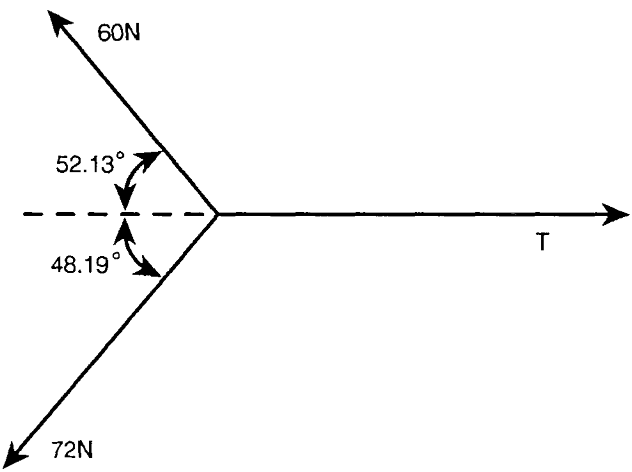
Statics

The first step in solving statics problems is to identify all forces acting on the body and draw a *free-body diagram*. A free-body diagram shows clearly the forces acting on the body, the relevant dimensions of the body, and the unknown forces or dimensions to be found.

Equilibrium of Forces

When bodies are at rest, the *vector sum* of those forces add to zero ($\sum F = 0$). It is easiest to resolve the forces acting on a body into their x-, y-, and z-components when working in rectangular coordinates or into the appropriate components when working in other coordinate systems (for example, spherical, polar, cylindrical, and so on).

Example. For the figure below, find the tension T in the cable that will balance the tension in the other two cables.



From the figure, it is clear that T must be such that the sum of forces in the x-direction is zero. T contributes no force in the y-direction since the sum of forces is already zero in that direction.

$$\sum F_x = 0 \rightarrow 60\cos 53.13° + 72\cos 48.19° = T$$

Calculate T as shown below.

Keys:	**Display:**	**Description:**
■ [MODES] {DG}		Puts calculator in Degrees mode.
60 [×] 53.13 [COS] [+] 72 [×] 48.19 [COS]		Calculates T.
[=]	83.9998	

Thus, the tension T must be about 84N for all the forces to be in equilibrium.

Moments and Couples

In addition to the vector sum of forces acting on the body, another concern is that the *moments* (sometimes called torques) acting on a body sum to zero as well. The moment $\boldsymbol{M_o}$ of a force $\boldsymbol{F}$ about a point O is given by

$$\boldsymbol{M_o} = \boldsymbol{r} \times \boldsymbol{F}$$

where $\boldsymbol{r}$ is a vector from O to *any point* on the line of action of $\boldsymbol{F}$. The operation above is a *cross product* and its result ($\boldsymbol{M_o}$) is a vector.* Since the sum of moments must be zero, this gives

$$\sum \boldsymbol{M_o} = 0$$

When two-dimensional problems are encountered, the formula for moments is particularly simple

$$M_o = Fd$$

where M_o is the magnitude of $\boldsymbol{M_o}$, F is the magnitude of $\boldsymbol{F}$, and d is the perpendicular distance between O and the line of action of $\boldsymbol{F}$. In two-

* For more information on cross products and vectors in general, see appendix B.

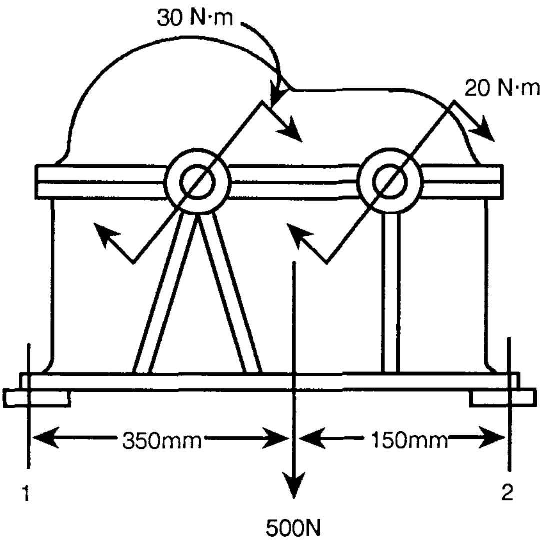
dimensional problems the moment can be thought of as acting either clockwise or counterclockwise about O; this is easily determined by inspection.

A *couple* is a special kind of moment that arises when two equal and opposite forces $\boldsymbol{F}$ and $-\boldsymbol{F}$ are separated by a perpendicular distance d. The moment due to a couple is given as $M = Fd$ where F is the magnitude of $\boldsymbol{F}$ and M is the magnitude of $\boldsymbol{M}$. It is sometimes convenient in three-dimensional problems to use the formula

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$$

where $\boldsymbol{r}$ is a vector connecting any point on the line of action of $-\boldsymbol{F}$ to any point on the line of action of $\boldsymbol{F}$. The important thing to remember about couples is that they produce a moment that is the *same* about *any* point O.

Example. Find the vertical forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ at each of the mountings 1 and 2 for the 500N gear reducer in the figure below.



If point 1 is chosen as the point about which moments will be calculated, notice that $\boldsymbol{F}_1$ contributes no moment (the line of action of $\boldsymbol{F}_1$ passes through point 1). The couple is the same about *any* point as noted previously and the sum of moments about 1 becomes

$$\sum M_1 = 0 \quad \rightarrow \quad 30+20+500(.350)=(.500)F_2$$

where the clockwise moments due to the two couples and the housing weight have been equated to the counterclockwise moment from the force at 2. To find F_2, follow the keystrokes below.

Keys:	**Display:**	**Description:**
30 [+] 20 [+] 500 [×] .35 [=] [÷] .5 [=]	450.0000	Calculates F_2.

Next, notice that the forces act only in the y-direction, giving the equation

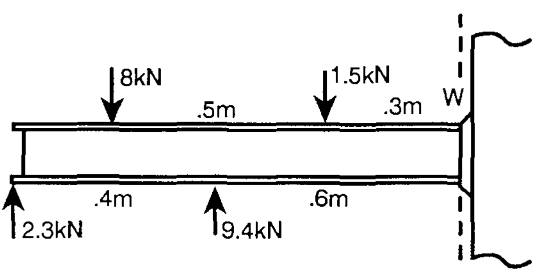
$$\sum F_y = 0 \quad \rightarrow \quad F_1+F_2=500$$

This can now be rearranged to find $F_1=500-F_2$ or $F_1=50\text{N}$.

Resultants

Sometimes the forces, moments, and couples acting on a body are replaced by an equivalent force that gives the same net effect on the body. This force is called the *resultant* of the system. No new equations are needed to find resultants; instead of all forces and moments adding to zero, they now add to a resultant force R and moment M.

Example. For the beam shown in the figure below, find a resultant force R and its distance d from the supporting weld at W.



The forces act only in the y-direction. Summing forces gives

$$\sum F_y = R \quad \rightarrow \quad 9.4 + 2.3 - 8 - 1.5 = R$$

To calculate R, follow the keystrokes below.

Keys:	**Display:**	**Description:**
9.4 [+] 2.3 [-] 8		Calculates R.
[-] 1.5 [=]	2.2000	

Thus, R is 2.2kN directed upward.

The moment about W is found as

$$\sum M_w = M \quad \rightarrow \quad 2.3(1) + 9.4(.6) - 8(.8) - 1.5(.3) = Rd = 2.2d$$

This gives

$$d = \frac{2.3(1) + 9.4(.6) - 8(.8) - 1.5(.3)}{2.2}$$

To calculate d, follow the keystrokes below.

Keys:	**Display:**	**Description:**
[(] 2.3 [+] 9.4 [×] .6 [-]		Calculates d.
8 [×] .8 [-] 1.5 [×] .3 [)]		
[÷] 2.2 [=]	0.4955	

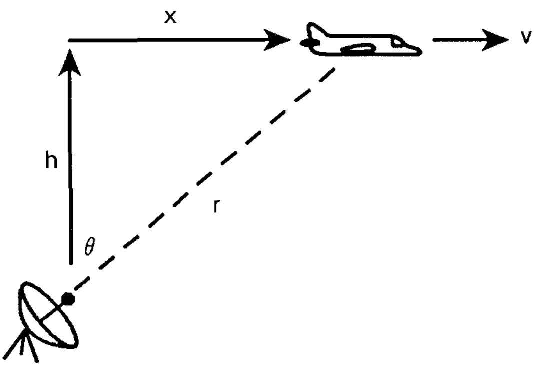
Thus, d is about 0.5 meters.

Dynamics

Motion

As a convenience, many texts use *dot notation* to indicate time derivatives of quantities. Thus, $\frac{dx}{dt}$ becomes $\dot{x}$ and $\frac{d^2x}{dt^2}$ becomes $\ddot{x}$. Many motion problems employ the basic relationships $v = \dot{x}$ and $a = \dot{v} = \ddot{x}$.

Example. A plane flying at velocity v is at a height h of 10km. It is being tracked by radar as shown in the figure below. If the angle θ is 56° and is increasing at a rate of 0.01 rad/sec, find v.



From the figure, x, h, and r form a right triangle and

$$\tan\theta = \frac{x}{h}$$

Noting that h is constant and differentiating with respect to time gives

$$\dot{\theta}\sec^2\theta = \frac{1}{h}\dot{x}$$

Some simple algebraic rearrangements yield

$$\dot{x} = v = h\,\dot{\theta} \sec^2\theta$$

Since θ, h, and $\dot{\theta}$ are known, v can be calculated as shown next.

Keys:	**Display:**	**Description:**
■ [MODES] {DG}		Sets Degrees mode.
10000 [×] .01 [×] 56 [COS] [1/x] ■ [x^2] [=]	319.7987	Calculates v.

Thus, the jet is flying at 319.7987 m/sec or about 1150 km/hr.

Work and Kinetic Energy

The work U done by a force $\mathbf{F}$ which acts on a body along a path $\mathbf{x}$ is given by

$$U = \int \mathbf{F} \cdot d\mathbf{x}$$

If the force is constant along the path and is in the same direction as the motion, this simplifies to

$$U = Fx$$

where F is the magnitude of the force, and x is the path length.

The kinetic energy T of a body is the energy due to its motion, given by

$$T = \frac{1}{2}mv^2$$

where m is the mass of the body in kg, v is the velocity of the body in m/sec, and T is the kinetic energy in joules. Work and kinetic energy are related by the *work-energy relation*

$$U = \Delta T$$

where ΔT is the change in kinetic energy as a body moves from the beginning to the end of its path.

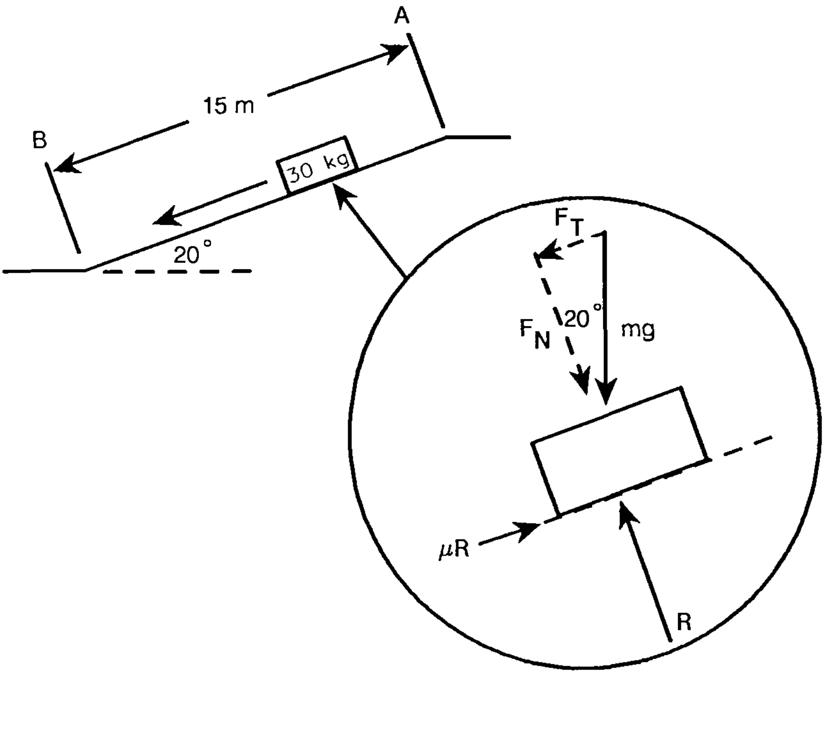
Friction

In many problems, the effects of friction must be included in the analysis. For a good number of surfaces a *coefficient of friction* μ is known and the simple relation

$$F = \mu N$$

is an adequate model. Here, F is the force due to friction and acts in a direction opposite to the motion of the body. N is the component of force between the two bodies which is normal to their mating surfaces.

Example. Find the velocity v of the 30 kg crate when it reaches point B. The crate has an initial velocity of 5 m/sec at point A and the coefficient of friction is 0.25.



The free body diagram is shown enlarged in the figure. The normal force R is equal and opposite to F_N, the normal component of the crate's weight. From the figure, this is

$$R = mg \cos 20°$$

Keys:	Display:	Description:
■ [MODES] {DG}		Puts calculator in Degrees mode.
30 [×] 9.80665 [×] 20 [COS] [=]	276.4571	Calculates R.
[×] .25 [=]	69.1143	Calculates μR, the frictional force.

The component of the crate's weight in the direction of motion (F_T) is given by

$$F_T = mg \sin 20°$$

Keys:	Display:	Description:
30 [×] 9.80665 [×] 20 [SIN] [=]	100.6222	Calculates F_T.

Now, the work done by gravity as the crate slides down the incline is found from the equation $U = Fx$. F is the total force in the direction of motion and is given by $F_T - \mu R$ for this problem.

Keys:	Display:	Description:
100.6222 [-] 69.1143 [=]	31.5079	Calculates F.
[×] 15 [=]	472.6185	Calculates U.

The change in kinetic energy is given by

$$\Delta T = \frac{1}{2} mg (v_B^2 - v_A^2)$$

And finally, v_B is found using the work-energy relation

$$U = \Delta T = \frac{1}{2} mg (v_B^2 - v_A^2) \quad \rightarrow \quad v_B = \left(\frac{2U}{mg} + v_A^2 \right)^{\frac{1}{2}}$$

Since U, g, m, and v_A are known, v_B is calculated as shown below.

Keys:	**Display:**	**Description:**
[(] 2 [×] 472.6185 [÷] [(] 30 [×] 9.80665 [)] [+] 5 ■ [x^2] [)] [$\sqrt{x}$]	5.3116	Calculates v_B.

Thus, the crate has a velocity of 5.3116 m/sec when it reaches point B.

8

Thermodynamics

This chapter contains the following examples from thermodynamics:

- Using linear interpolation with thermodynamic tables.
- Pressure calculations using a manometer.
- Finding the change in quality of saturated water.
- Heat transfer using the first law of thermodynamics.
- Net change in entropy for a process using the second law of thermodynamics.

To work the examples in this chapter, you may find the thermodynamic tables available in most texts helpful.

Linear Interpolation

When the values needed in a thermodynamics calculation fall between successive entries in a table, linear interpolation is used. In other words, it is usually assumed that table entries are spaced close enough together so that negligible error is introduced when a linear relationship is assumed between any two adjoining entries. Your HP-22S performs linear interpolation calculations easily using the {$\hat{x}$} and {$\hat{y}$} keys in the STAT menu.

The steps for performing linear interpolation are as follows:

1. Clear any previously stored statistical data by pressing ■ [CLEAR] {Σ}.
2. Enter the first x,y-pair as x_1 [INPUT] y_1 [Σ+].
3. Enter the second x,y-pair as x_2 [INPUT] y_2 [Σ+].
4. Enter the known x-value in the display and press [STAT] {L.R.} {$\hat{y}$}.

These keystrokes draw a straight line through the two x,y-pairs that you input and then find a y-value ($\hat{y}$ in the menu) lying in this line given an x-value. You can also find an x-value given a y-value in a completely analagous fashion to that given above.*

Example. Superheated water vapor is in a rigid, sealed container at 163°C. If the pressure of the water vapor is 70kPa, what is its specific volume v in m^3/kg?

From tables at 70kPa, the following information is available:

$$160°\text{C} \quad \rightarrow \quad v = 2.841\frac{\text{m}^3}{\text{kg}}$$

$$180°\text{C} \quad \rightarrow \quad v = 2.975\frac{\text{m}^3}{\text{kg}}$$

* For additional information on using the statistical functions of your HP-22S, see chapter 5 of your owner's manual.

Use the method for linear interpolation outlined previously with

$$x_1 = 160$$
$$y_1 = 2.841$$
$$x_2 = 180$$
$$y_2 = 2.975$$
$$x = 163$$

Keys:	Display:	Description:
■ [CLEAR] {Σ}		Clears all previously stored statistical data.
160 [INPUT] 2.841 [Σ+]	n=1.0000	Stores the first x,y-pair.
180 [INPUT] 2.975 [Σ+]	n=2.0000	Stores the second x,y-pair.
163 [STAT] {L.R.} {$\hat{y}$}	$\hat{y}$=2.8611	Calculates y given x.

Thus, at 163°C, v is approximately 2.8611 m^3/kg.

Pressure

Pressure is defined as the normal component of force per unit area. The SI unit for measuring pressure is the pascal (Pa) and is given by

$$1\text{Pa} = 1\text{N/m}^2.$$

A device commonly used to measure the difference in pressure ΔP is a liquid-filled u-shaped tube called a *manometer*. The relation

$$\Delta P = \rho g L$$

is used where ρ is the density of the liquid in the tube, g is the acceleration due to gravity, and L is the difference in height between the liquid levels on the two sides of the tube.

Example. A pressure drop across an orofice in a natural gas line is measured with a mercury manometer. At 20° C, mercury has a density of 13,546 kg/m^3. If $g = 9.80665$ m/sec^2, what difference in height L corresponds to a 20kPa pressure drop?

The HP-22S equation

P=D×G×L

is used to solve this problem where L is the height difference, G is the acceleration due to gravity, D is the liquid density, and P is the change in pressure. Type this equation into your list of equations and follow the next set of keystrokes.

Keys:	**Display:**	**Description:**
■ [SOLVE] {L}	D G L P P?value	Selects L, prompts for P.
20000 [INPUT]	P=20,000.0000 D?*value*	Stores P, prompts for D.
13546 [INPUT]	D=13,546.0000 G?*value*	Stores D, prompts for G.
9.80665 [INPUT]	G=9.8067 L=0.1506	Stores G, calculates L.

Thus, a height difference of 150.6 mm corresponds to a 20 kPa pressure drop.

Specific Volume, Density, and Quality

Density is a familiar concept to most people. The reciprocal quantity, specific volume v, is used extensively in thermodynamic calculations. Specific volume has units of m^3/kg. The quality x of a substance is defined as the ratio of vapor mass m_g to total mass (vapor mass plus liquid mass m_f) of a substance when it is in the saturation state (exists in both liquid and vapor states).

$$x = \frac{m_g}{m_g + m_f}$$

This can also be found using the equation

$$v = (1-x)v_f + xv_g$$

where v_f and v_g are the specific volumes of the liquid and vapor, respectively, and v is the total specific volume.

Example. A rigid, sealed vessel contains saturated water at 150° C with a quality of 0.47. The saturated water is heated to a temperature of 180° C. Find the new quality.

Since the vessel is sealed, no mass can leave or enter it. Since the vessel is rigid, no change in volume can occur. Thus, the specific volume does not change as the temperature changes. If the system at 150° is denoted as state 1 and the system at 180° as state 2, then the total specific volume in state 1 is found as

$$v_1 = (1-x_1)v_{f_1} + x_1 v_{g_1}$$

From tables,

$$v_{f_1} = .001091$$
$$v_{g_1} = .3928$$
$$v_{f_2} = .001127$$
$$v_{g_2} = .19405$$

Substitute these values into the equation above to obtain v_1 as shown on the next page.

Keys:	Display:	Description:
[(] 1 [−] .47 [)] [×] .001091 [+] .47 [×] .3928 [=]	0.1852	Calculates v_1.

Now, find x_2, the quality at state 2:

$$v_1 = v_2 = (1 - x_2)v_{f_2} + x_2 v_{g_2} \quad \rightarrow \quad \frac{v_1 - v_{f_2}}{v_{g_2} - v_{f_2}} = x_2$$

To obtain x_2, follow the keystrokes below.

Keys:	Display:	Description:
[(] .1852 [−] .001127 [)] [÷] [(] .19405 [−] .001127 [)] [=]	0.9541	Calculates x_2.

The quality at 180°C is, therefore, 0.9541.

The First Law of Thermodynamics

The first law of thermodynamics is often referred to as the law of the conservation of energy. When a system changes from an initial state 1 to a final state 2, the 1st law may be expressed as

$$\int_1^2 \delta Q - \int_1^2 \delta W = \Delta\mathrm{KE} + \Delta\mathrm{PE} + \Delta\mathrm{U}$$

where

$\int_1^2 \delta Q$ is the heat transferred to the system

$\int_1^2 \delta W$ is the work done by the system as it goes from state 1 to 2

ΔKE is the change in the kinetic energy of the system
ΔPE is the change in the potential energy of the system
ΔU is the change in the internal energy of the system

Notice that δQ and δW are used as integrands instead of *d* Q and *d* W. The δ indicates that Q and W are *path functions*. That is, they depend on the process used in moving from state 1 to state 2, not just the initial and final states.

The internal energy U is often expressed as a per unit mass quantity u where $u = \mathrm{U}/m$. Like specific volume, the internal energy is included in most thermodynamic tables and exhibits the same relationship with quality:

$$u = (1-x)u_f + xu_g$$

Example. Assume that 3kg of water is in the vessel of the previous example. Using the first law of thermodynamics, calculate the heat in joules added to the system to raise its temperature from 150°C to 180°C.

Here, the first law equation becomes

$$\text{heat added} = \int_1^2 \delta Q = \Delta\mathrm{KE} + \Delta\mathrm{PE} + \Delta\mathrm{U} + \int_1^2 \delta W$$

No work is done by the system as it moves from state 1 to state 2 and changes in kinetic and potential energies are negligible. This gives

$$\int_1^2 \delta Q = \Delta U = m(u_2-u_1)$$

From tables,

$$u_{f_1} = 631.68 \text{ kJ/kg}$$
$$u_{g_1} = 2559.5 \text{ kJ/kg}$$
$$u_{f_2} = 762.09 \text{ kJ/kg}$$
$$u_{g_2} = 2583.7 \text{ kJ/kg}$$

Since the qualities in both states 1 and 2 were found in the previous example, the equations

$$u_1 = (1-x_1)u_{f_1}+x_1u_{g_1}u_2 = (1-x_2)u_{f_2}+x_2u_{g_2}$$

are employed. Follow the keystrokes on the next page to find u_1 and u_2 recalling that $x_1 = 0.47$ and $x_2 = 0.9541$.

Keys:	**Display:**	**Description:**
[(] 1 [-] .47 [)] [×] 631.68 [+] .47 [×] 2559.5 [=]	1,537.7554	Calculates u_1.
[STO] A	A=1,537.7554	Stores u_1 in A.
[(] 1 [-] .9541 [)] [×] 762.09 [+] .9541 [×] 2583.7 [=]	2,500.0881	Calculates u_2.
[-] [RCL] A [=]	962.3327	Calculates u_2-u_1.
[×] 3 [=]	2,886.9981	Calculates $\Delta U = m\,\Delta u$.

Thus, about 2887 kJ of heat are transferred to the water to increase its temperature.

The Second Law of Thermodynamics

The second law of thermodynamics says, in essence, that construction of a perfectly efficient machine or process is impossible. For example, if 1000 kJ of heat are transferred to a steam boiler, the boiler will *always* deliver less than 1000 kJ of work. The second law is described quantitatively using a property called *entropy* S. As a per-unit mass quantity, the entropy is $s = S/m$. Entropy is usually expressed in units of $\frac{\text{kJ}}{\text{kg}-\text{K}}$ where K is degrees Kelvin. Like v and u, entropy s appears in most thermodynamic tables.

Using S, the second law demands that

$$\Delta S_{net} = \Delta S_{sys} + \Delta S_{surr} = \Delta S_{sys} - \frac{1}{T_0}\int_1^2 \delta Q \geq 0$$

where ΔS_{sys} is the change in entropy of the system, ΔS_{surr} is the change in entropy of the surroundings, and T_0 is the temperature of the surroundings in Kelvin.

Another property, *enthalpy* H, is included in its per-unit mass form h in most tables. The units of h are generally kJ/kg. For the special case in which a quasi-equilibrium process occurs at constant pressure

$$\int_1^2 \delta Q = \Delta H$$

This says that the heat transfer is equal to the change in enthalpy for constant pressure, quasi-equilibrium processes.

Example. A constant pressure of 500kPa is maintained in a rigid, sealed vessel containing 10kg of saturated water. At this pressure, the saturation temperature is 151.86°C. The water vapor is condensed at constant temperature until it is all liquid and heat is transferred to the surrounding air. The ambient air temperature is 20°C (293.15 K). Calculate the net change in entropy.

To solve this problem, first find the entropy change of the system (the water in the vessel). From tables at 500kPa and 151.86°C,

$$s_g = 6.8213 \frac{\text{kJ}}{\text{kg}\cdot\text{K}} \text{ (entropy of vapor)}$$

$$s_f = 1.8607 \frac{\text{kJ}}{\text{kg}\cdot\text{K}} \text{ (entropy of liquid)}$$

And the system's entropy change is found as

$$\Delta S_{sys} = m\,\Delta s_{sys} = m\,(s_2 - s_1) = m\,(s_f - s_g)$$

Keys:	**Display:**	**Description:**
10 [×] [(] 1.8607 [−] 6.8213 [)] [=]	−49.6060	Calculates ΔS_{sys}.

Now the change in the entropy of the surroundings is calculated as

$$\Delta S_{surr} = -\frac{1}{T_0}\int_1^2 \delta Q$$

However, this is a constant pressure process and

$$\int_1^2 \delta Q = \Delta H \quad \rightarrow \quad \Delta S_{surr} = -\frac{m\,\Delta h}{T_0} = \frac{m\,(h_1 - h_2)}{T_0} = \frac{m\,(h_g - h_f)}{T_0}$$

From tables,

$$h_g = 2748.7 \text{ kJ/kg}$$
$$h_f = 640.23 \text{ kJ/kg}$$

Follow the keystrokes below to calculate ΔS_{surr}.

Keys:	**Display:**	**Description:**
10 [×] [(] 2748.7 [−] 640.23 [)] [=]	21,084.7000	Calculates $m\,\Delta h$.
[÷] 3 [=]	7,028.2333	Divides by T_0 (in Kelvin) to calculate ΔS_{surr}.

The final answer is obtained from

$$\Delta S_{net} = \Delta S_{sys} + \Delta S_{surr} \ .$$

Keys:	**Display:**	**Description:**
49.6060 [+/−] [+] ■ [LAST] [=]	6,978.6273	Calculates ΔS_{net}.

The net change in entropy is positive as the second law requires and is equal to 6,978.6273 kJ/K.

9

Electrical Fundamentals

This chapter contains examples from electrical fundamentals:

- Current in a DC circuit.
- Finding a y-equivalent circuit.
- Steady-state current in an AC circuit.
- Cut-off frequency of a low-pass filter.

DC Circuit Analysis

DC circuit analysis will almost always consist of circuits with only resistors, dependent sources, and independent sources. When inductors appear in a DC circuit, they are treated as short circuits (no resistance), and when capacitors appear, they are treated as an open circuit (infinite resistance). The formulas for DC analysis are:

- The passive sign convention requires that if the voltage across a circuit element is positive to negative (a voltage drop) in the direction of current flow through the element, a positive sign is used in the expression relating voltage and current for the element. Otherwise a negative sign is used.
- Kirchoff's voltage law demands that the sum of voltage drops around any closed loop in a circuit is zero.
- Kirchoff's current law demands that the algebraic sum of currents at any node in a circuit is zero.
- Ohm's Law:

$$\frac{V}{I} = R$$

- The power in a general circuit component:

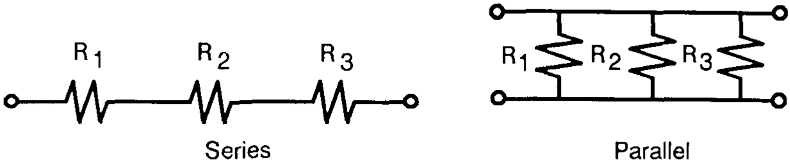
$$P = IV$$

- The power in a resistor:

$$P = I^2 R$$

where V is the voltage in volts, R is the resistance in ohms, I is the current in amperes, and P is the power in watts.

Often, resistors will be connected in series or in parallel as shown below.



The formulas for the equivalent resistance of resistors connected this way are given below.

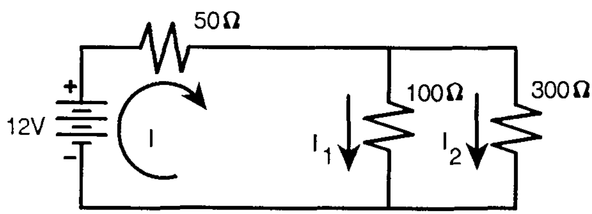
- Series

$$R_{eq} = \sum_{k=1}^{N} R_k$$

- Parallel

$$\frac{1}{R_{eq}} = \sum_{k=1}^{N} \frac{1}{R_k}$$

Example. Find the current I_1 in the 100Ω resistor for the circuit below.



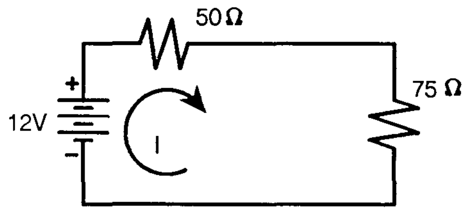
To solve this problem, first find the total current I by finding the equivalent resistance of the circuit. The 100Ω and 300Ω resistors are connected in parallel. the formula for parallel resistance becomes

$$R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

Calculate R_{eq} as shown below.

Keys:	**Display:**	**Description:**
100 [1/x] [+] 300 [1/x] [=] [1/x]	75.0000	Calculates R_{eq}.

The circuit can now be redrawn as shown.



The total current I can be found easily since the 50 Ω and 75 Ω resistors are connected in series giving a total resistance of $50 + 75 = 125$ Ω. The total current I is then found using Ohm's Law of $I = V/R$.

Keys:	**Display:**	**Description:**
12 [÷] 125 [=] [STO] I	I=0.0960	Calculates and stores I.

From the first figure, observe that I flows through the 50 Ω resistor and splits into the branches containing the 100 Ω and 300 Ω resistors. For two reistors R_1 and R_2 in parallel, the current divider equation may be used where

$$I_1 = \frac{I\ R_2}{R_1 + R_2}$$

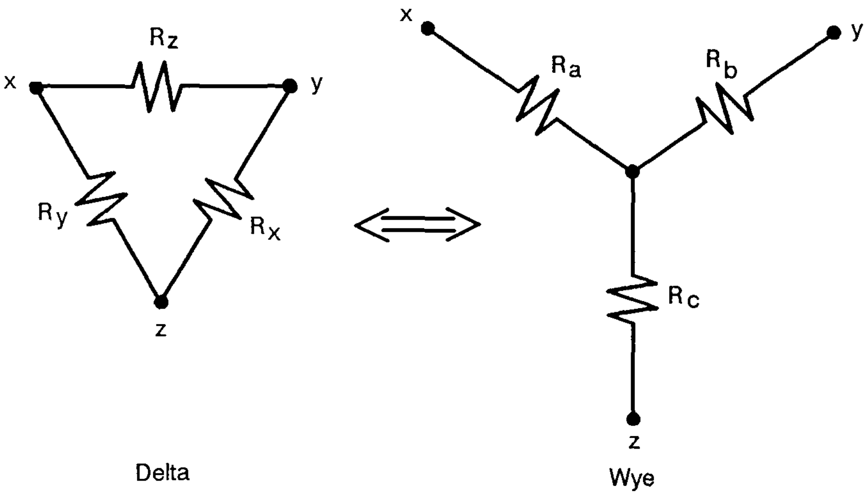
Here, $R_1 = 100\,\Omega$ and $R_2 = 300\,\Omega$. Use this equation to find I_1.

Keys:	Display:	Description:
[RCL] I [×] 300 [÷] [(]		Calculates I_1.
100 [+] 300 [)] [=]	0.0720	

Thus, 72 mA flow through the 100 Ω resistor.

Δ-Y Conversions

Resistors can be connected in a Δ configuration (sometimes called a π configuration) or in a Y configuration (sometimes called a T configuration) as shown below.



Circuit analysis is sometimes simplified by transforming a Δ configuration to a Y configuration or vice-versa. The formulas below show how to do this.

Δ to Y:

$$R_a = \frac{R_y R_z}{R_x + R_y + R_z}$$

$$R_b = \frac{R_x R_z}{R_x + R_y + R_z}$$

$$R_c = \frac{R_x R_y}{R_x + R_y + R_z}$$

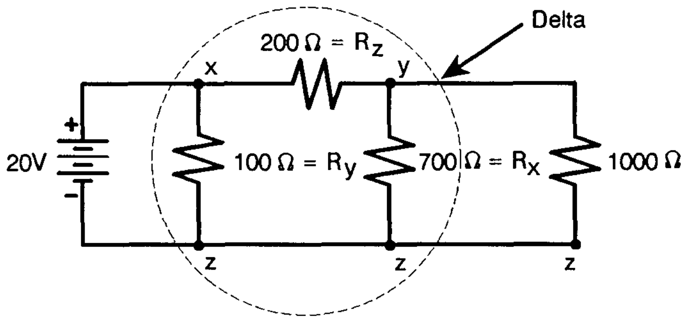
Y to Δ:

$$R_x = \frac{R_b R_c}{R_a + R_b + R_c}$$

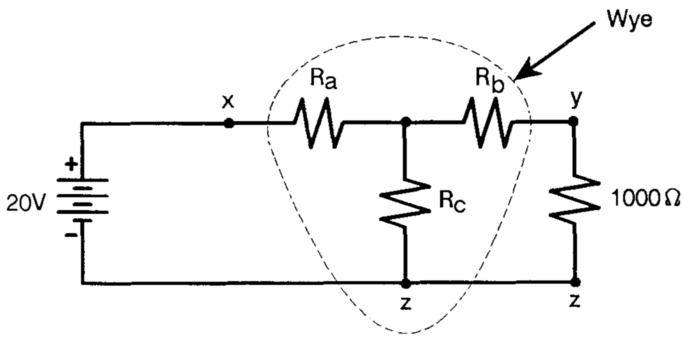
$$R_y = \frac{R_a R_c}{R_a + R_b + R_c}$$

$$R_z = \frac{R_a R_b}{R_a + R_b + R_c}$$

Example. Find the Y equivalent circuit for the Δ circuit below.



The equivalent Y circuit will be configured as shown.



First, calculate R_a.

Keys:	Display:	Description:
100 [×] 200 [÷] [(] 100 [+] 200 [+] 700 [)] [STO] T [=]	20.0000	Calculates R_a and stores $R_x + R_y + R_z$ in T (for total).

Next, calculate R_b.

Keys:	Display:	Description:
700 [×] 200 [÷] [RCL] T [=]	140.0000	Calculates R_b.

Finally, calculate R_c.

Keys:	Display:	Description:
700 [×] 100 [÷] [RCL] T [=]	70.0000	Calculates R_c.

Thus, $R_a = 20\Omega$, $R_b = 140\Omega$, and $R_c = 70\Omega$.

Sinusoidal Steady-State Analysis

When a circuit consisting of resistors, inductors, and capacitors is driven by a sinusoidal voltage or current, the *steady-state* response is also sinusoidal with the same frequency as the driving voltage or current.

Impedance

In AC steady-state analysis, Kirchoff's voltage and current laws still apply, but Ohm's law is replaced by a more general expression that allows for other circuit elements besides resistors. Instead of R, a quantity Z called the *impedance* of the element is used to give the formula $V = IZ$. Notice that V and I are used to denote *phasor* quantities which will be explained in a moment. The impedances of linear circuit elements are as follows:

- Capacitors

$$Z_C = \frac{1}{j\omega C}$$

- Inductors

$$Z_L = j\omega L$$

- Resistors

$$Z_R = R$$

Here C is the capacitance in Farads, L is the inductance in Henrys, R is the resistance in Ohms, ω is the radian frequency of the driving source, and j is the unit imaginary number.* Remember that these impedances are only valid when the driving function is *sinusoidal* of radian frequency ω. Notice that the impedances listed above are both real and imaginary numbers. Hence, their combinations yield complex numbers in general. Often, it is most convenient to manipulate complex numbers such as

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* It is customary to use j as the unit imaginary number instead of i since i is often used to denote current.

impedances in their exponential form.* A shorthand notation called *angle notation* is often used where a complex number of magnitude A and angle θ is denoted by $A \angle \theta$. This notation will be used in this book.

The formulas for computing equivalent impedances are the same as for resistors.

- Series

$$Z_{eq} = \sum_{k=1}^{N} Z_k$$

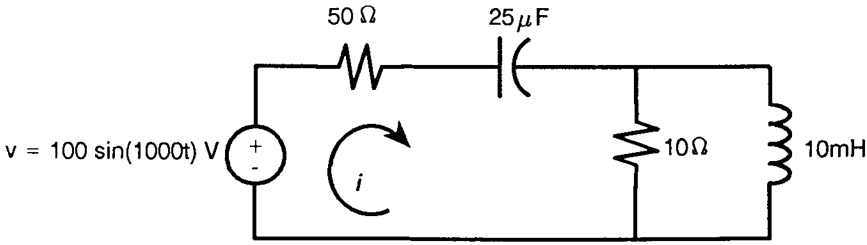
- Parallel

$$\frac{1}{Z_{eq}} = \sum_{k=1}^{N} \frac{1}{Z_k}$$

Phasors

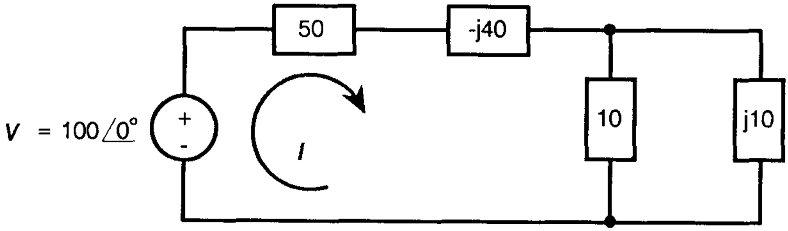
It can be shown that if a sinusoidal voltage or current $M \sin(\omega t + \phi)$ is replace by its *phasor transform* $Me^{j\phi}$, then this complex number (the phasor) can be used in the equations relating voltage and current such as $V = IZ$. Thus, phasors are a shorthand way of manipulating sinusoidal functions as complex numbers. Phasors are denoted by boldface type to distinguish them from other complex numbers.

Example. Find the steady-state current i in the circuit below.



* For more information on complex numbers see appendix A.

From the figure, the frequency of the driving voltage is $\omega = 1000$, its magnitude is 100 volts, and its phase is 0°. Thus it is denoted by the phasor $V = 100\angle 0°$. To find i, first redraw the circuit as shown where each of the impedances have been calculated using the appropriate formula and $\omega = 1000$.



The impedances in parallel are converted to an equivalent impedance using a special case of the parallel impedance formula when there are only two impedances

$$Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Here, $Z_1 = 10$ and $Z_2 = j\,10$ which gives

$$Z_{eq} = \frac{j\,100}{10 + j\,10}$$

In angle notation, the numerator is written as $100\angle 90°$. To express the denominator in angle notation, it is necessary to convert it to polar form using the rectangular-to-polar function.

Keys:	**Display:**	**Description:**
■ [MODES] {DG}		Sets Degrees mode.
10 ■ [x COORD]	x=10.0000	Stores the real part of the number.
10 ■ [y COORD]	y=10.0000	Stores the imaginary part of the number.

■ [RADIUS]	r=14.1421	Magnitude of denominator.
■ [ANGLE]	θ=45.0000	Angle of denominator.

This gives the impedance in angle notation as

$$Z_{eq} = \frac{100\underline{/90^\circ}}{14.1421\underline{/45^\circ}}$$

It is easiest to multiply and divide complex numbers when they are in exponential form. To divide, simply divide magnitudes and subtract angles; to multiply, multiply magnitudes and add angles. Here division is needed. Thus,

$$Z_{eq} = \frac{j\,100}{10+j\,10} = \frac{100\underline{/90^\circ}}{14.1421\underline{/45^\circ}} = \frac{100}{14.1421}\underline{/90^\circ - 45^\circ} = 7.0711\underline{/45^\circ}$$

This equivalent impedance in polar form is now converted to rectangular form so it can be added easily to the impedances of the capacitor and 50Ω resistor.

Keys:	**Display:**	**Description:**
7.0711 ■ [RADIUS]	r=7.0711	Stores the magnitude of the impedance.
45 ■ [ANGLE]	θ=45.0000	Stores the angle of the impedance.
■ [x COORD]	x=5.0000	Calculates the real part of the impedance.
■ [y COORD]	y=5.0000	Calculates the imaginary part of the impedance.

This gives $Z_{eq} = 5 + j\,5$. Adding this to the other impedances in series with it gives $Z_{total} = 55 - j\,35$. This is converted to polar form and angle notation.

Keys:	**Display:**	**Description:**
55 ■ [x COORD]	x=55.0000	Stores the real part of the total impedance.
35 [+/-] ■ [y COORD]	y= −35.0000	Stores the imaginary part of the total impedance.
■ [RADIUS]	r=65.1920	Magnitude of total impedance.
■ [ANGLE]	θ= −32.4712	Angle of total impedance.

To arrive at the final result, the equation $I = \dfrac{V}{Z_{total}}$ is used. Thus,

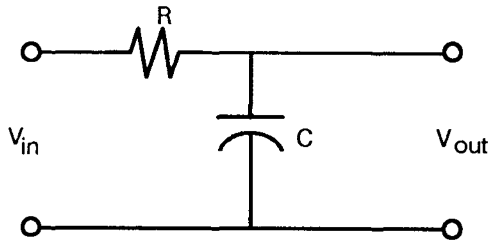
$$I = \frac{V}{Z_{total}} = \frac{100\underline{/0^\circ}}{65.1920\underline{/-32.4712^\circ}} = \frac{100}{65.1920}\underline{/0-(-32.4712^\circ)} = 1.5339\underline{/32.4712^\circ}$$

which gives $i = 1.5339\sin(1000t + 32.4712^\circ)$ amperes.

Frequency Response

Frequency response is described as how a circuit responds in the steady state to sinusoids of different frequencies.

Example. The circuit below is a simple low-pass filter.



Intuition verifies this by noting that at frequency 0 the capacitor allows no current flow and all the input voltage V_{in} appears across the capacitor. At infinite frequency the capacitor behaves like a short circuit, and no voltage appears across it giving $V_{out} = 0$. Thus, the filter effectively keeps very high frequencies from appearing at the output but allows low frequencies to appear as an output voltage. The *cut-off frequency* is that frequency at which $|V_{out}| = \frac{1}{\sqrt{2}} |V_{in}|$. If $R = 1000\,\Omega$ and $C = 50\,\mu$F, find the cut-off frequency for the above filter.

Using the *voltage divider* equation, the input voltage can be related to the output voltage as

$$V_{out} = \frac{V_{in} Z_C}{Z_R + Z_C}$$

Using the formulas for Z_R and Z_C gives

$$V_{out} = \frac{V_{in} \frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{V_{in}}{j\omega RC + 1}$$

Comparing this equation with the defining equation for the cut-off frequency gives

$$\frac{1}{\sqrt{2}} = \left| \frac{1}{j\omega RC + 1} \right| = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

This last step was accomplished by using the formula for finding the magnitude of a complex number, given on page 143 in appendix A.

Key in the HP-22S equation

1÷SQRT(2)=1÷SQRT(1+SQ(W×R×C))

Then follow these keystrokes:

Keys:	**Display:**	**Description:**
■ [SOLVE] {W}	C R W R?*value*	Selects W, prompts for R.
1000 [INPUT]	R=1,000.0000 C?*value*	Stores *R*, and prompts for *C*.
50 ■ [E] [-] 6 [INPUT]	C=0.0001 W=20.0000	Stores *C* and calculates *W*.

The cut-off frequency is at 20 radians per second or $2\pi\omega = f = 125.67$ Hz.

10

Advanced Equation Writing Techniques

More about EVAL and SOLVE

Chapter 6 of your owner's manual describes several differences between [EVAL] and [SOLVE]. These are repeated here, with some additions.

1. When an equation is of the form:

   unknown variable = *algebraic expression*

   [EVAL] evaluates the right side of the equation and sets the *unknown variable* equal to that value. When ■ [SOLVE] is pressed, the calculator allows you to select which variable is solved.

2. When the equation is an *algebraic expression* (no equal sign occurs), [EVAL] prompts for the value of each variable in the expression and then calculates the value of the expression. When ■ [SOLVE] is pressed, the calculator allows you to select which variable will be solved for and then solves for this such that the expression is equal to zero.

3. When an equation contains more than one variable on the left-hand side of the equal sign, pressing [EVAL] automatically switches the HP-22S to [SOLVE].

4. When an equation is of the form shown in item 1 above, and the *algebraic expression* contains the *unknown variable*, [EVAL] evaluates the *algebraic expression* using the *current* value of the *unknown variable*. This result is then assigned as the *new* value of the *unknown variable*. Pressing ■ [SOLVE], on the other hand, finds a mathematically "correct" solution for the equation.

The use of [EVAL] in item 4 has many practical applications, as shown next.

Using New and Old Values with EVAL

Many problems involve computing a new value of a variable based on the old value of that variable. These are often referred to as *recursive* problems. The next example shows how a simple recursive problem might arise and gives an HP-22S equation that models the problem.

Example 1: A Traffic Counter. To determine if enough traffic passes through an intersection to justify purchasing a stoplight, a city installs an electronic counter to record the number of vehicles that use the intersection in a month. If the number exceeds 5,000, a stoplight is to be purchased.

Here, the counter must increment its current value by 1 each time a vehicle passes. Thus, the current value of the counter is used to find the next value, creating a recursive problem. The HP-22S equation

A=A+1

will behave like a counter when [EVAL] is pressed. Notice that A appears all by itself on the left side of the equation and *also* appears on the right side. Thus, the conditions described in item 4 on page 131 apply. Each time [EVAL] is pressed, the right side of the equation (A+1) is evaluated using the current value of A. This result is then assigned as the new value of A. Key the above equation into your list of equations, press [INPUT], and follow the keystrokes below.

Keys:	Display:	Description:
[EVAL]	A?*value*	Displays current value of A, prompts for new value.
0 [INPUT]	A=0.0000 A=1.0000	Enters the value 0 for A, and then computes the new value for A using this value.

EVAL	A?1.0000	Displays current value of A, prompts for new value of A.
INPUT	A=1.0000 A=2.0000	Keeps the current value for A (1), and then computes the new value for A using this value.

This process can be repeated indefinitely to continue incrementing A by 1. Notice that if ■ SOLVE is pressed instead of EVAL, the calculator displays NO ROOT FND. The equation $A = A + 1$ has no solution as can be noted by subtracting A from each side.

Example 2: Computing a Summation. A summation can also be thought of as a recursive problem with the next value of the sum equal to the current value plus the next term. Consider the series

$$\sum_{j=0}^{N} \frac{1}{2^j}$$

It is known that this sum approaches 2 as $N \to \infty$. Find the smallest value for N such that the sum is within 1% of 2.

This involves finding the smallest N such that the sum is greater than or equal to 1.98. The HP-22S equation

```
A=A+.5^J
```

can be used to solve this problem. Notice again that the conditions of item 4 on page 131 are met and thus, EVAL can be used to add the next term of the sum (.5^J) to the current value of the sum (A). This result is then assigned as the new value of A. Key in this equation, press INPUT, and follow the next set of keystrokes.

Keys:	Display:	Description:
EVAL	A?*value*	Displays current value of A, prompts for new value of A.
0 INPUT	A=0.0000 J?*value*	Enters 0 for A since you want the running total of the sum to start with zero. The current value of J is then displayed and the calculator prompts for a new value.
0 INPUT	J=0.0000 A=1.0000	Enters 0 for J and calculates a new value for A. This is the value of the sum when the first term is added ($N=0$).
EVAL	A?1.0000	Displays current contents of A and prompts for new value of A.
INPUT	A=1.0000 J?0.0000	Keeps current contents of A since this is the value of the sum to which you want to add the next term. Displays the current value of J, and prompts for a new value.
1 INPUT	J=1.0000 A=1.5000	The value of J is assigned its next value of 1, and the next term in the sum is added to give the new value of A.

You should continue this process by pressing EVAL, retaining the current contents of A, and incrementing J by 1 until a result of $A=1.9844$ is obtained. This occurs when $J=6$. Thus, the smallest value for N satisfying the given requirement is $N=6$.

The previous two examples could have been done with storage arithmetic, but this would result in more keystrokes. In example 2, for instance, each term of the series could be computed with the HP-22S equation Q=.5^J where J is incremented each time Q is calculated. Each time Q is found, the result could be accumulated in variable A using [STO] [+] A. This is clearly more cumbersome than the method used in example 2. In general, use of [EVAL] in the manner presented here can often replace the manual keystrokes used in storage arithmetic.

Example 3: Computing the Covariance. In chapter 8 of your owner's manual you are given an equation for finding the covariance of x,y-data. There you are required to use storage arithmetic ([STO] [+] C). Using the [EVAL] technique of this book, the HP-22S equation

C=C+(X−x̄)×(Y−ȳ)÷n

eliminates the need for manual storage arithmetic.*

Find the covariance of the data in the example of your owner's manual. You can use the same keystrokes as shown in the owner's manual except now each time you press [EVAL], the calculator will prompt for C. Store a zero in C initially. Thereafter, simply press [INPUT] after each [EVAL] to retain the current value of C. Also, be sure to disregard the [STO] C, [STO] [+] C, and [RCL] C keystrokes. You should obtain the same result of $C = 1.08$ but with eight less keystrokes.

* The variables $\bar{x}, \bar{y}$, and n are found in the [STAT] menu.

Using Control Variables

When a variable is used in an HP-22S equation to control the form of the solution, it is referred to as a *control variable*. An example of this is the variable J in the quadratic equation in the LIBRARY of your HP-22S. Here, J simply serves the role of a $\pm$ sign and allows both real roots to be computed using the same equation.

This technique is quite useful and eliminates having to store two separate equations in memory which differ only by a minus sign. This not only saves memory, but it also saves keystrokes by eliminating the need to load a different equation. An extension of this technique involves using a control variable at two different places in an equation. This is introduced next.

Multiplication by Zero

When you are required to find results from two expressions that use the same variables, a simple and useful technique is that of *multiplication by zero*. An HP-22S equation of the form

unknown variable = (*control expression A* × *expression 1*) + (*control expression B* × *expression 2*)

can use a control variable in the two *control expressions* to display either the value of *expression 1* or *expression 2* when the *unknown variable* is calculated. This is accomplished by causing *control expression A* to be 1 while *control expression B* is 0, and vice-versa. A specific example will illustrate this idea.

Example: Using Multiplication by Zero. Suppose you want to compute both $a + b^c$ and $\ln(a - b + c)$ for specific values of a, b, and c.

The HP-22S equation

```
J×(A+B^C)+(1-J)×LN(A-B+C)
```

will accomplish this by using the control variable J. Notice that when $J = 1$, the natural log expression is multiplied by 0 and the other expression is multiplied by 1. When $J = 0$, the opposite result occurs. Key this equation into your list of equations, and press [INPUT] to find the values of the two expressions for $a = 2$, $b = 3$, and $c = 4$.

Keys:	**Display:**	**Description:**
[EVAL]	J?*value*	Prompts for value of J and displays current value.
0 [INPUT]	J=0.0000 A?*value*	Stores J and prompts for A.
2 [INPUT]	A=2.0000 B?*value*	Stores A and prompts for B.
3 [INPUT]	B=3.0000 C?*value*	Stores B and prompts for C.
4 [INPUT]	C=4.0000 EXPR=1.0986	Stores C and calculates the value of the second expression $\ln(a+b-c)$.
[EVAL]	J?0.0000	Prompts for value of J.
1 [INPUT]	J=1.0000 A?2.0000	Stores J and prompts for A.
[INPUT]	A=2.0000 B?3.0000	Stores A and prompts for B.
[INPUT]	B=3.0000 C?4.0000	Stores B and prompts for C.
[INPUT]	C=4.0000 EXPR=83.0000	Stores C and calculates the value of the first expression $a+b^c$.

On page 22, multiplication by zero was used to solve a practical problem that arises often in science and engineering, that of finding complex roots for a quadratic equation.

Storing Constants

When certain constants are used repeatedly, you may wish to store them in your list of equations. For example, to store the acceleration due to gravity (9.80665 m/sec^2) in a variable called G, use the HP-22S equation

```
G=9.80665
```

When you press [EVAL], this value is stored in G; [RCL] G can be used wherever this constant is needed in your calculations. A number of constants can be stored in this way and used as needed.

If you wish to store the acceleration due to gravity in both SI and British Engineering System units, the two equations

```
G=9.80665
G=32.1740
```

can be used. The second equation gives g in ft/ sec^2. By making either one of these equations the current equation in your list of equations, pressing [EVAL] stores the appropriate value. You may want to adopt as a memory aid a convention of always placing the constant with SI units in your list of equations so that it appears *before* the same constant with British Engineering System units.

A Helpful Hint

This chapter has introduced several tools that give you the ability to write more powerful HP-22S equations. Of course, you will not want to enter all the equations you encounter into the list of equations in your HP-22S. In many cases it is faster simply to work the equation through manually. However, when you must use an equation repeatedly, or when you are finding roots, you will want to write an appropriate HP-22S equation. When doing so, be sure to look for applications of the techniques presented here. Judicious use of [EVAL], control variables, and multiplication by zero will save you time by reducing keystrokes; they also utilize the memory available for storing equations in a more efficient manner.

A

Additional Algebra Formulas

Exponent Properties

For the properties listed here, let a, b, and x be real numbers.

The Common Base Property:

$$x^a x^b = x^{a+b}$$

The Power of a Power Property:

$$(x^a)^b = x^{ab}$$

The Product of Base Property:

$$(ab)^x = a^x b^x$$

The Negative Exponent Property:

$$a^{-x} = \frac{1}{a^x}$$

The Fractional Exponent Property:

$$x^{\frac{a}{b}} = \sqrt[b]{x^a}$$

Logarithm Properties

By definition, the logarithm to the base a of b is the number x where

$$x = \log_a b \text{ if and only if } a^x = b .$$

Here, $a > 0$, $a \neq 1$, and $b > 0$.

Frequently used properties of logarithms are as follows with Q, P, and a positive numbers, $a \neq 1$, and x any real number:

The Power of a Logarithm Property:

$$a^{\log_a Q} = Q$$

The Logarithm of a Product Property:

$$\log_a (PQ) = \log_a P + \log_a Q$$

The Logarithm of a Fraction Property:

$$\log_a \frac{P}{Q} = \log_a P - \log_a Q$$

The Logarithm of a Power with Common Base Property:

$$\log_a a^x = x$$

The Logarithm of an Inverse Property:

$$\log_a \frac{1}{P} = -\log_a P$$

The Logarithm of a Power with Different Base Property:

$$\log_a P^x = x \log_a P$$

The Logarithm of Unity Property:

$$\log_a 1 = 0$$

The Change of Base Formula:

$$\log_a b = \frac{\log_c b}{\log_c a}$$

where $a > 0$, $c > 0$, $a \neq 1$, $c \neq 1$, and $b > 0$.

Complex Numbers

Conversion Formulas

The Rectangular-to-Polar Conversion:

$$r = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1} \frac{b}{a}$$

The Polar-to-Rectangular Conversion:

$$a = r \cos\theta \qquad b = r \sin\theta$$

Useful Results of Complex Numbers. For the following results, let $z = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$ with a, b, c, and d real.

Equality of complex numbers:

$$z = w \quad \text{if and only if} \quad a = c \quad \text{and} \quad b = d$$

Addition of complex numbers:

$$z + w = (a + \mathrm{i}b) + (c + \mathrm{i}d) = (a + c) + \mathrm{i}(b + d)$$

Subtraction of complex numbers:

$$z - w = (a + \mathrm{i}b) - (c + \mathrm{i}d) = (a - c) + \mathrm{i}(b - d)$$

Multiplication of complex numbers in rectangular form:

$$zw = (a + \mathrm{i}b)(c + \mathrm{i}d) = (ac - bd) + \mathrm{i}(ad + bc)$$

Division of complex numbers in rectangular form:

$$\frac{z}{w} = \frac{a + \mathrm{i}b}{c + \mathrm{i}d} = \frac{(a + \mathrm{i}b)(c - \mathrm{i}d)}{(c + \mathrm{i}d)(c - \mathrm{i}d)} = \frac{(ac + bd) + \mathrm{i}(bc - ad)}{c^2 + d^2}$$

Euler's Identity:

$$e^{\pm \mathrm{i}\theta} = \cos\theta \pm \mathrm{i}\sin\theta$$

where θ is in radians.

Complex conjugate z^* of z (where $z = a + \mathrm{i}b = re^{\mathrm{i}\theta}$):

$$z^* = a - \mathrm{i}b = re^{-\mathrm{i}\theta}$$

Multiplication of complex numbers in polar form:

$$zw = r_z e^{\mathrm{i}\theta_z} r_w e^{\mathrm{i}\theta_w} = r_z r_w e^{\mathrm{i}(\theta_z + \theta_w)}$$

Division of complex numbers in polar form:

$$\frac{z}{w} = \frac{r_z e^{\mathrm{i}\theta_z}}{r_w e^{\mathrm{i}\theta_w}} = \frac{r_z}{r_w} e^{\mathrm{i}(\theta_z - \theta_w)}$$

Inverse of a complex number:

$$\frac{1}{z} = \frac{a - \mathrm{i}b}{a^2 + b^2} = \frac{1}{r} e^{-\mathrm{i}\theta}$$

Logarithm of a complex number:

$$\ln(z) = \ln(re^{\mathrm{i}\theta}) = \ln(r) + \mathrm{i}(2\pi k + \theta)$$

where θ is in *radians*, $k = 0, 1, 2, 3, \ldots$, and $k = 0$ is the *principle value* of the logarithm function.

Real power p of a complex number:

$$z^p = [re^{\mathrm{i}\theta}]^p = r^p e^{\mathrm{i}p\theta} = r^p [\cos p\theta + \mathrm{i}\sin p\theta]$$

where θ is in *radians*.

Integer roots of a complex number:

$$z^{1/n} = r^{1/n}\, e^{i\theta/n} = r^{1/n}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right]$$

where θ is in *radians* and n is any positive integer. The n nth roots of z can be found by substituting $k = 0, 1, 2, ..., n - 1$. This result is known as *De Moivre's Theorem*.

Complex power of a complex number:

$$z^w = e^{\ln(z^w)} = e^{w \ln z} = e^{(c \ln r_z - d\theta_z)}\, e^{i(d \ln r_z + c\theta_z)}$$

where θ is in *radians* and Euler's Identity is used to expand the second exponential.

Complex form of the sine:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Complex form of the cosine:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

These results use r_z, θ_z, r_w, and θ_w as the magnitudes and angles of z and w respectively. Be sure to use θ in *radians* when computing logarithms and powers of complex numbers. Since the real numbers are a sub-set of the complex numbers, these results are valid for real z and w also.

Polynomial Equations

A polynomial equation of degree n is expressed in the form

$$y = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x^1 + a_0$$

where the a_j's are real constants and n is any positive integer. It is a well-known result of algebra that an equation of degree n has exactly n roots. A root is a value of x for which y equals zero.

By factoring the above equation, the n roots can be readily seen as

$$y = a_n(x - r_n)(x - r_{n-1}) \cdots (x - r_2)(x - r_1)$$

where the r_j's are roots.

For example, the equation $y = x^2 + 2x - 3$ can be factored to give $y = (x + 3)(x - 1)$. The roots are evident immediately as $x = -3$ and $x = 1$.

Sometimes, roots are not distinct, but rather *repeated* as in the equations

$$y = x^3 - 3x - 2 = (x + 1)^2(x - 2)$$

and

$$y = x^4 + 8x^3 + 24x^2 + 32x + 16 = (x + 2)^4$$

Nature of Roots for Polynomials

- If an equation is *odd* in degree, it has at least one real root. If it has more than one real root, there must be an odd number of them.
- If an equation is *even* in degree, its roots may consist only of complex conjugate pairs; in other words, the equation may have *no* real roots. If the equation does have real roots, there must be an even number of them.
- If a complex root is found, its conjugate is also a root.
- When a real root is found, use polynomial division to factor out the term containing the root. This will leave a polynomial of one less degree, which will generally be easier to locate roots for.
- When a complex root is found, find the complex conjugate corresponding to the first root. Multiply the two factors together which contain these two roots. This will give a real quadratic factor of the form $(x^2 + bx + c)$. Dividing the original equation by this quadratic factor will result in a polynomial of two less degrees which will generally be easier to locate roots for.

Quadratic, Cubic, and Quartic Equations. Exact formulas for finding the roots of quadratic, cubic, and quartic equations are:

Quadratic equation $ax^2 + bx + c = 0.$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cubic equation $x^3 + ax^2 + bx + c = 0$

Let

$$K = \frac{3b - a^2}{9}$$

$$L = \frac{9ab - 27c - 2a^3}{54}$$

$$M = \left[L + \sqrt{K^3 + L^2}\right]^{\frac{1}{3}}$$

$$N = \left[L - \sqrt{K^3 + L^2}\right]^{\frac{1}{3}}$$

then

$$x_1 = M + N - \frac{a}{3}$$

$$x_2 = -\frac{1}{2}(M + N) - \frac{a}{3} + \frac{\sqrt{3}}{2}\mathrm{i}(M - N)$$

$$x_3 = -\frac{c}{x_1 x_2} \quad (\text{if } x_2 \text{ is complex, } x_3 = x_2^*)$$

Quartic equation $x^4 + ax^3 + bx^2 + cx + d = 0.$

If z_1 is a real root of the cubic equation

$$z^3 - bz^2 + (ac - 4d)z + (4bd - c^2 - a^2d) = 0$$

then the four roots of the quartic equation are the four roots of

$$y^2 + \frac{1}{2}\left[a \pm \sqrt{a^2 - 4b + 4z_1}\right]y + \frac{1}{2}\left[z_1 \mp \sqrt{z_1^2 - 4d}\right] = 0$$

which can be found using the quadratic formula.

Factorials

If n is a positive integer greater than 0, then

$$n! = (1)(2)(3)...(n)$$

and, by definition

$$0! = 1$$

The Binomial Formula

For $n = 1, 2, 3, \cdots$

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n}b^n$$

where

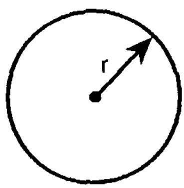
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

B

Additional Geometry and Trigonometry Formulas

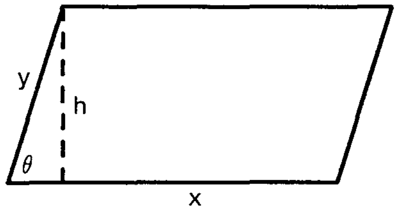
Selected Geometric Figures

Circle of radius r:



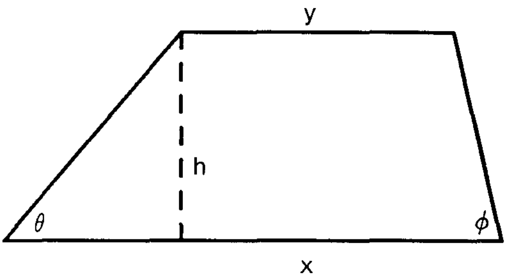
Area $= \pi r^2$
Circumference $= 2\pi r$

Parallelogram:



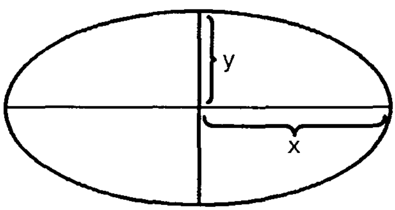
Area $= xh = xy \sin\theta$
Perimeter $= 2y + 2x$

Trapezoid:



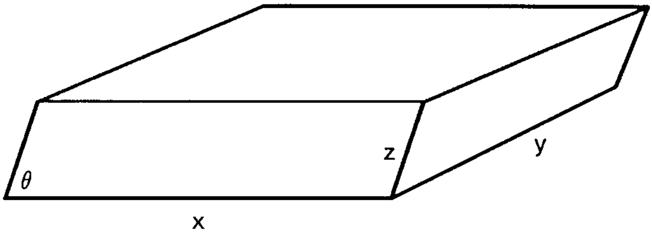
$$\text{Area} = \frac{1}{2}h\,(x+y)$$
$$\text{Perimeter} = x + y + h\,(\csc\theta + \csc\phi)$$

Ellipse:



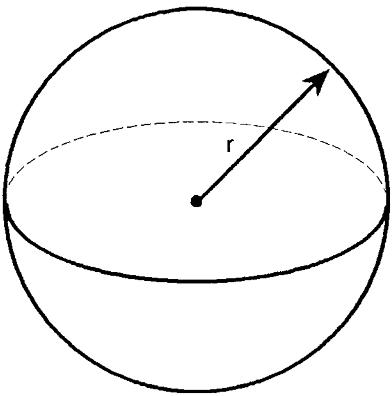
$$\text{Area} = \pi xy$$
$$\text{Perimeter} = 4x\int_0^{\frac{\pi}{2}} \sqrt{1-\frac{x^2-y^2}{x^2}\sin^2\theta}\;d\theta \approx 2\pi\sqrt{\frac{x^2+y^2}{2}}$$

Parallelepiped:



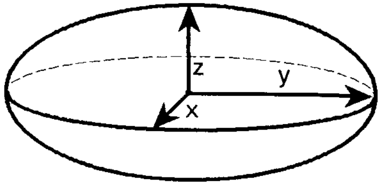
$$\text{Volume} = xyz \sin\theta$$
$$\text{Surface Area} = 2xy + 2xz \sin\theta + 2zy$$

Sphere of radius r:



$$\text{Volume} = \frac{4}{3}\pi r^3$$
$$\text{Surface Area} = 4\pi r^2$$

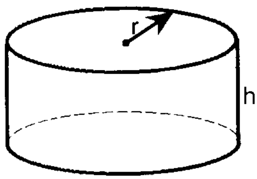
Ellipsoid:



$$\text{Volume} = \frac{4}{3}\pi xyz$$

$$\text{Surface Area}^* = \begin{cases} 2\pi xy\left[\dfrac{y}{x} + \dfrac{x}{x^2 - y^2}\sin^{-1}\left(\dfrac{x^2 - y^2}{x}\right)\right] & x > y \\ 2\pi xy\left[\dfrac{y}{x} + \dfrac{x}{y^2 - x^2}\ln\left(\dfrac{y + y^2 - x^2}{x}\right)\right] & x < y \end{cases}$$

Cylinder:



$$\text{Volume} = \pi r^2 h$$
$$\text{Surface Area} = 2\pi rh + 2\pi r^2$$

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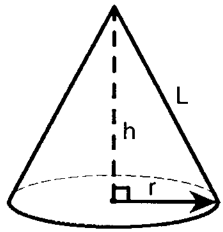
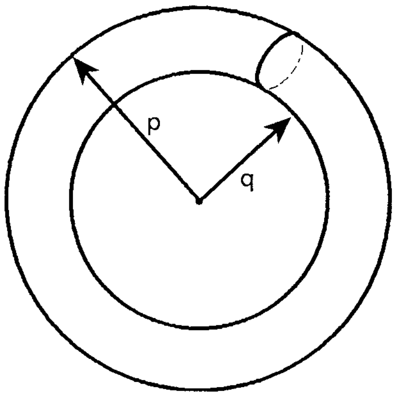
* This is the surface area of an ellipsoid of revolution formed when an ellipse in the *xy*-plane is revolved about the *x*-axis.

Right Circular Cone:



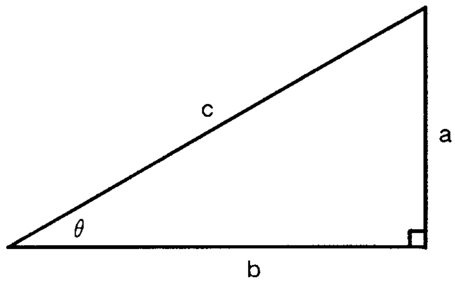
$$\text{Volume} = \frac{1}{3}\pi r^2 h$$
$$\text{Surface Area} = \pi r L + \pi r^2$$

Torus:



$$\text{Volume} = \frac{1}{4}\pi^2(p+q)(p-q)^2$$
$$\text{Surface Area} = \pi^2(p^2-q^2)$$

Definitions of Trigonometric Formulas



For the right triangle above, the trigonometric functions of angle θ are as follows:

- sine:

$$\sin\theta = \frac{a}{c}$$

- cosine:

$$\cos\theta = \frac{b}{c}$$

- tangent:

$$\tan\theta = \frac{a}{b}$$

- cosecant:

$$\csc\theta = \frac{1}{\sin\theta} = \frac{c}{a}$$

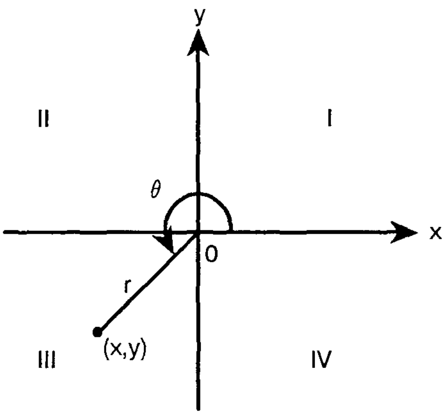
- secant:

$$\sec\theta = \frac{1}{\cos\theta} = \frac{c}{b}$$

- cotangent:

$$\cot\theta = \frac{1}{\tan\theta} = \frac{b}{a}$$

Definitions for Angles Greater Than 90°



For the angle θ in any quadrant of the figure above, the following definitions apply:

$$r^2 = x^2 + y^2 \quad (\textit{Pythagorean formula})$$

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y}$$

$$\sec\theta = \frac{r}{x}$$

$$\cot\theta = \frac{x}{y}$$

Degrees and Radians Conversions

There are 2π radians in a complete circle giving the equality

$$2\pi \text{ radians} = 360°$$

or

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$

$$1° = \frac{\pi}{180} \text{ radians}$$

Your HP-22S performs these conversions easily using the {→RAD} and {→DEG} keys found in the D←→RAD menu.

Basic Trigonometric Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\csc^2\theta - \cot^2\theta = 1$$

Cofunction Formulas

$$\sin\theta = \cos(\pi/2-\theta)$$

$$\cos\theta = \sin(\pi/2-\theta)$$

$$\cot\theta = \tan(\pi/2-\theta)$$

$$\tan\theta = \cot(\pi/2-\theta)$$

$$\sec\theta = \csc(\pi/2-\theta)$$

$$\csc\theta = \sec(\pi/2-\theta)$$

Functions of Negative Angles

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\cot(-\theta) = -\cot\theta$$

$$\csc(-\theta) = -\csc\theta$$

Addition Formulas

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Subtraction Formulas

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Double Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Half Angle Formulas

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

$$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}$$

Sums, Differences, and Products

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = 2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$$

Inverse Trigonometric Functions and Principle Values

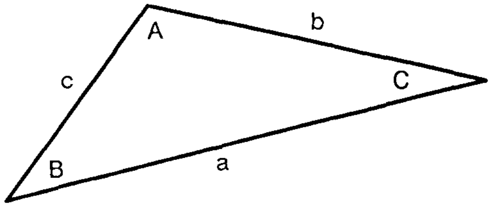
An inverse trigonometric function involves finding the angle y whose trigonometric function has the value x. For example, $y = \tan^{-1}x$ means "find the angle y whose tangent is x."

The inverse trigonometric functions are multivalued. For example, $y = \sin^{-1}0$ is solved for $y = n\pi$ where $n = 1, 2, 3, \ldots$. Rather than an infinite family of solutions, a *principle value* is often desired for the inverse trigonometric functions. They are as follows:

$\theta \geq 0°$	$\theta \leq 0°$
$0° \leq \sin^{-1}\theta \leq 90°$	$-90° \leq \sin^{-1}\theta < 0°$
$0° \leq \cos^{-1}\theta \leq 90°$	$90° < \cos^{-1}\theta \leq 180°$
$0° \leq \tan^{-1}\theta < 90°$	$-90° < \tan^{-1}\theta < 0°$
$0° < \cot^{-1}\theta \leq 90°$	$90° < \cot^{-1}\theta < 180°$
$0° \leq \sec^{-1}\theta < 90°$	$90° < \sec^{-1}\theta \leq 180°$
$0° < \csc^{-1}\theta \leq 90°$	$-90° \leq \csc^{-1}\theta < 0°$

Laws of Sines, Cosines, and Tangents

These formulas fold for any plane triangle ABC as shown in the next figure.



Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Law of Tangents:

$$\frac{a+b}{a-b} = \frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}$$

C

Additional Linear Algebra Formulas

Basic Properties of Vectors

Assume that **A**, **B**, and **C** are vectors, and r and s are scalars,

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$$

$$\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$$

$$rs\,\mathbf{A}=r(s\,\mathbf{A})=s(r\,\mathbf{A})$$

$$(r+s)\mathbf{A}=r\,\mathbf{A}+s\,\mathbf{A}$$

$$r(\mathbf{A}+\mathbf{B})=r\,\mathbf{A}+r\,\mathbf{B}$$

$$\mathbf{A}=a_1\,\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$$

where **i**, **j**, and **k** are unit vectors in the x, y, and z directions of a rectangular coordinate system. For the magnitude of **A** (| **A** |):

$$|\mathbf{A}|=\sqrt{a_1^2+a_2^2+a_3^2}$$

A unit vector (magnitude 1) in the direction of **A** is given by

$$\frac{\mathbf{A}}{|\mathbf{A}|}$$

Vector Multiplication

For the formulas below,

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$
$$\mathbf{B} = b_1 i + b_2\mathbf{j} + b_3\mathbf{k},$$

and

$$\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

are vectors.

Dot Product

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$$

where θ is the angle between **A** and **B**.

Distributive Property of Dot Product

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$

Cross Product

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

This vector is perpendicular to both **A** and **B**.

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$

$$\mathbf{A}\times(\mathbf{B} + \mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$$

$$|\mathbf{A}\times\mathbf{B}| = |\mathbf{A}|\,|\mathbf{B}|\sin\theta$$

This is the area of a parallelogram with adjacent sides **A** and **B** with angle θ between them.

More Vector Product Formulas

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The absolute value of this result is the volume of a parallelepiped with sides **A**, **B**, and **C**.

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$$

Angle θ between vectors

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|}$$

Component of **A** in the direction of **B**

$$\mathrm{comp}_{\mathbf{B}}\mathbf{A} = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|^2}\mathbf{B}$$

The magnitude of this vector is $|\mathbf{A}|\cos\theta$.

Component of **A** orthogonal to **B**

$$\mathbf{A} - \mathrm{comp}_{\mathbf{B}}\mathbf{A} = \mathbf{A} - \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|^2}\mathbf{B}$$

Matrix Formulas

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

then

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \ldots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} + b_{m1} & a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

Multiplying a matrix **A** by a scalar k is done by multiplying each element of **A** by k.

$\mathbf{I}_n$ is the *identity matrix* of order n. This matrix is square with dimensions $n \times n$. All entries are 0 except for 1's along the main diagonal. As an example,

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Two matrices **C** and **D** can be multiplied *only* if the number of columns in **C** equals the number of rows in **D**. Let

$$\mathbf{C} = \left[c_{ij}\right] \text{ be an } m \times n \text{ matrix}$$

and

$$\mathbf{D} = \left[d_{ij}\right] \text{ be an } n \times p \text{ matrix}$$

The matrix product

$$\mathbf{E} = \mathbf{CD} = \left[e_{ij}\right] \text{ is an } m \times p \text{ matrix}$$

where the element e_{ij} in the ith row and jth column of **E** is $e_{ij} = c_{i1}b_{1j} + c_{i2}b_{2j} + \cdots + c_{in}b_{nj}$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., p$.

The inverse $\mathbf{A}^{-1}$ of an $n \times n$ matrix **A** is the matrix with the property $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$.

When $\mathbf{A}^{-1}$ does not exist, **A** is a *singular* matrix. If $\mathbf{A}^{-1}$ does exist, **A** is nonsingular.

If det $\mathbf{A} = 0$, then **A** is singular.

The transpose $\mathbf{A}^T$ of **A** is formed by interchanging the rows and columns of **A**. For example, row 1 of **A** becomes column 1 of $\mathbf{A}^T$, row 2 of **A** becomes column 2 of $\mathbf{A}^T$, and so forth. If $\mathbf{A}^T = \mathbf{A}$, **A** is *symmetric*.

Inverse of a 3×3 matrix:

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}} \begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

Solving a System of Linear Equations

A set of n linear equations in n unknowns can be written as:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\ &\;\;\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= c_n \end{aligned}$$

where each a_{ij} is a constant coefficient, each c_i is a constant, and each x_j is an unknown.

This system of equations is written compactly in matrix form as

$$\mathbf{A}x = \mathbf{C}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{n1} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \text{(coefficient matrix)}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \qquad \text{(unknown vector)}$$

and

$$\mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{bmatrix} \qquad \text{(constant vector)}$$

The unknowns are then found as

$$x = \mathbf{A}^{-1}\mathbf{C}$$

Cramer's Rule

Cramer's Rule can also be used to solve a system of simultaneous linear equations. To use Cramer's Rule, write the system in matrix form as shown above. Cramers Rule gives

$$x_n = \frac{\det \mathbf{A}_n}{\det \mathbf{A}}$$

where $\mathbf{A}_n$ is the matrix formed by replacing the nth column of **A** with the constant vector **C**.

Properties of Determinants

Determinants are defined only for square matrices.

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\det\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = A\begin{vmatrix} E & F \\ H & I \end{vmatrix} - B\begin{vmatrix} D & F \\ G & I \end{vmatrix} + C\begin{vmatrix} D & E \\ G & H \end{vmatrix}$$

If two rows or two columns of a matrix are interchanged, reverse the sign of the determinant.

If all the elements of one row of a matrix **C** are multiplied by the same constant k, then the determinant of this new matrix is k det**A**.

If all the elements of one row of a matrix **C** are multiplied by the same constant k and then added to the corresponding elements of another row of **C**, the determinant of this new matrix is unchanged.

$$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$$

D

Additional Calculus Formulas

Derivatives

Definition

The *derivative* of $f(x)$ with respect to x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x)$ is also denoted by $\frac{df(x)}{dx}$.

Basic Properties of Derivatives

For the following, w, y, and z are functions of x, e is the natural logarithm base, and k and n are constants.

$$\frac{d}{dx}(k) = 0$$

$$\frac{d}{dx}(kx) = k$$

$$\frac{d}{dx}(kx^n) = nkx^{n-1}$$

$$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$$

$$\frac{d}{dx}(w \pm y \pm z \pm ...) = \frac{dw}{dx} \pm \frac{dy}{dx} \pm \frac{dz}{dx} \pm \cdots$$

$$\frac{d}{dx}(yz) = y\frac{dz}{dx} + z\frac{dy}{dx}$$

$$\frac{d}{dx}(y/z) = \frac{z(dy/dx) - y(dz/dx)}{z^2}$$

The *Chain Rule* of differentiation

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

Derivatives of Exponential and Logarithmic Functions

$$\frac{d}{dx}e^y = e^y\frac{dy}{dx}$$

$$\frac{d}{dx}\ln(y) = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{d}{dx}k^y = \frac{d}{dx}e^{y\ln k} = k^y \ln k\frac{dy}{dx}$$

$$\frac{d}{dx}y^z = \frac{d}{dx}e^{z\ln y} = e^{z\ln y}\frac{d}{dx}(z\ln y) = zy^{z-1}\frac{dy}{dx} + y^z \ln y\frac{dz}{dx}$$

Derivatives of Trigonometric Functions

Here, all angles must be in radians.

$$\frac{d}{dx}\sin y = \cos y\frac{dy}{dx}$$

$$\frac{d}{dx}\cos y = -\sin y\frac{dy}{dx}$$

$$\frac{d}{dx}\tan y = sec^2 y \frac{dy}{dx}$$

$$\frac{d}{dx}cot\ y = -csc^2 y \frac{dy}{dx}$$

$$\frac{d}{dx}csc\ y = -csc\ y\ cot\ y \frac{dy}{dx}$$

$$\frac{d}{dx}sec\ y = sec\ y\ \tan y \frac{dy}{dx}$$

$$\frac{d}{dx}\sin^{-1}y = \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} \qquad \left[-\frac{\pi}{2} < \sin^{-1}y < \frac{\pi}{2}\right]$$

$$\frac{d}{dx}\cos^{-1}y = \frac{-1}{\sqrt{1-y^2}}\frac{dy}{dx} \qquad \left[0 < \cos^{-1}y < \pi\right]$$

$$\frac{d}{dx}\tan^{-1}y = \frac{1}{1+y^2}\frac{dy}{dx} \qquad \left[-\frac{\pi}{2} < \tan^{-1}y < \frac{\pi}{2}\right]$$

$$\frac{d}{dx}cot^{-1}y = \frac{-1}{1+y^2}\frac{dy}{dx} \qquad \left[0 < cot^{-1}y < \pi\right]$$

$$\frac{d}{dx}sec^{-1}y = \frac{1}{|u|\sqrt{y^2-1}}\frac{dy}{dx}$$

$$\frac{d}{dx}csc^{-1}y = \frac{-1}{|u|\sqrt{y^2-1}}\frac{dy}{dx}$$

Partial Derivatives

If $f(u,v)$ is a function of two variables u and v, the partial derivative of f with respect to u is written as $\frac{\partial f}{\partial u}$ and is obtained by differentiating f with respect to u and treating v as a constant. $\frac{\partial f}{\partial v}$ is defined analagously.

Definition of Differentials

If $y = f(x)$ then the differential of y is dy where

$$dy = f'(x)\Delta x$$

Thus, to find a differential dy, simply find $\frac{dy}{dx}$ and multiply by Δx.

The differential dy can be interpreted as the *approximate* change in y due to a change Δx in x. Think of dy as the change that would result in y if it were to change along the fixed line $f'(x)$ as x changes.

The true change in y is Δy and is based on y changing along the curve $y = f(x)$, not the fixed line $f'(x)$. For small changes in x, the approximation given by the differential is a good one. The *total differential* of $f(u, v, \ldots)$ where f is a function of several variables is given by

$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \cdots$$

Integrals

If $\frac{dy}{du} = f(u)$, then $y(u)$ must be a function with derivative $f(u)$. Computing an indefinite integral involves finding $y(u)$ given $f(u)$. Loosely speaking then, integration is equivalent to "undoing" differentiation. Often, $y(u)$ is called the *antiderivative* of $f(u)$. The indefinite integration procedure outlined above is given by the equation

$$y(u) = \int f(u)du$$

Since the derivative of a constant C is zero, the function $y(u) + C$ also has $f(u)$ as its derivative. Thus, given a function $f(u)$, there is an infinite family of antiderivatives which differ only by an arbitrary constant. This constant is called a constant of integration. In the indefinite integrals shown here, the constant of integration is omitted but implied.

Properties of Integration

Here, k is a constant and y, u, and v are functions of x.

$$\int k\,dx = kx$$

$$\int (u \pm v \pm \cdots)dx = \int u\,dx \pm \int v\,dx \pm \cdots$$

$$\int ky(x)dx = k\int y(x)dx$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Integration by Substitution

$$\int f(y(x))y'(x)dx = \int f(u)du$$

where $u = y(x)$ and $du = y'(x)dx$.

$$\int \frac{dx}{x} = \ln|x|$$

If the integrand is a function of more than one variable, treat all variables execept the variable of integration as constants. For example, when integrating the equation

$$\int f(x,u,v)dx$$

treat the variables u and v as constants.

Definite Integrals

A definite integral is of the form

$$\int_a^b f(x)dx$$

It can be interpreted as the area underneath $f(x)$ between the points $x = a$ and $x = b$. a and b are called the limits of integration and $f(x)$ is called the integrand. A definite integral is computed using the result as follows:

$$\int_a^b f(x)dx = G(x)\Big|_{x=a}^{x=b} = G(b)-G(a)$$

where $G(x)$ is the antiderivative of $f(x)$.

Properties of Definite Integrals

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

Integration by Substitution

$$\int_a^b f(y(x))y'(x)dx = \int_{y(a)}^{y(b)} f(u)du$$

where $u = y(x)$ and $du = y'(x)dx$.

The rules given for indefinite integrals also apply to definite integrals.

Improper Integrals

An improper integral is a definite integral in which one or both of the limits of integration are infinite, or in which the integrand becomes infinite or undefined at some point in the interval. Indefinite integrals must then be computed as limits as shown below.

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b f(x)dx$$

If c is a point between z and b at which the integrand becomes infinite or undefined, use the formula

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)dx + \lim_{\varepsilon \to 0} \int_{c+\varepsilon}^b f(x)dx$$

Differentiating Definite Integrals

The result below is often called *Leibnitz's Rule*.

$$\frac{d}{du}\int_{p(u)}^{q(u)} f(x,u)dx = \int_{p(u)}^{q(u)} \frac{\partial f(x,u)}{\partial u}dx + f(q(u),u)\frac{dq}{du} - f(p(u),u)\frac{dp}{du}$$

Transformation of Multiple Integrals

Sometimes a change of coordinate systems can make evaluation of a multiple integral easier. This is done as follows:

$$\iiint G(x,y,z)dxdydz = \iiint H(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

This determinant is called the *Jacobian* of the transformation. Notice that the absolute value of the Jacobian is used in the transformation.

Some frequently used transformation results are given below.

Rectangular to Polar Coordinates

$$\iint G(x,y)dxdy = \iint H(r,\theta)rdrd\theta$$

where

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Rectangular to Cylindrical Coordinates

$$\iiint G(x,y,z)\,dx\,dy\,dz = \iiint H(r,\theta,z)\,r\,dr\,d\theta\,dz$$

where

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Rectangular to Spherical Coordinates

$$\iiint G(x,y,z)\,dx\,dy\,dz = \iiint H(r,\theta,\phi)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

where

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Numerical Integration

Simpson's Rule is widely used to approximate definite integrals. This is due to its simplicity, good results, and ease of implementation. Simpson's Rule essentially divides the area to be integrated into an even number of subintervals and interpolates a quadratic polynomial to $f(x)$ at the top of each subinterval.

For integrals of the form

$$\int_a^b f(x)\,dx$$

the approximation for an even number of subintervals ($2n$) is given by:

$$S_{2n} = \frac{(b-a)/2n}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \ldots + 2f_{2n-2} + 4f_{2n-1} + f_{2n}\right]$$

This approximation is valid if the integral meets the following conditions:

1. The limits of integration are finite.
2. For all $a \le x \le b$, $f(x)$ is both finite and defined.

Taylor Series

Many functions of interest can be expressed as an infinite series called a *Taylor Series*. Provided that a function has derivatives of all orders on an interval containing the point a, its Taylor series is given as

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(a) \frac{(x-a)^k}{k!}$$

This series is a polynomial in $(x - a)$ that is constructed in such a way that the polynomial and all of its derivatives match the function $f(x)$ and all its derivatives at the point $x = a$. Often it is desirable to approximate a function with its *nth degree Taylor polynomial*. This is done by replacing the infinite upper limit for k in the sum with a finite value n. When a function $f(x)$ is to be approximated near a point a, the Taylor polynomial should be built around the term $(x - a)$. For the special case of $a = 0$, the series is sometimes called a *Maclaurin series*.

Maclaurin series for several common functions are given below:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad -\infty < x < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad -\infty < x < \infty$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad -\infty < x < \infty$$

$$\ln x = 2\left\{\left(\frac{x-1}{x+1}\right) + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right\} \qquad x > 0$$

L'Hopital's Rule for Limits

If

$$\lim \frac{g(x)}{h(x)} = \frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}$$

then

$$\lim \frac{g'(x)}{h'(x)} = \lim \frac{g(x)}{h(x)}$$

where lim denotes $\lim\limits_{x \to a}$, $\lim\limits_{x \to a^+}$, $\lim\limits_{x \to a^-}$, $\lim\limits_{x \to +\infty}$, or $\lim\limits_{x \to -\infty}$.

E

Selected Tables

Prefixes Used with Metric Quantities

Prefix:	Multiple:	Abbreviation:
exa-	10^{18}	E
pecta-	10^{15}	P
tera-	10^{12}	T
giga-	10^{9}	G
mega-	10^{6}	M
kilo-	10^{3}	k
hecto-	10^{2}	h
deka-	10	da
deci-	10^{-1}	d
centi-	10^{-2}	c
milli-	10^{-3}	m
micro-	10^{-6}	μ
nano-	10^{-9}	n
pico-	10^{-12}	p
femto-	10^{-15}	f
atto-	10^{-18}	a

Fundamental SI Units

Quantity:	Unit:	Abbreviation:
Electric current	ampere	A
Length	meter	m
Luminous intensity	candela	cd
Mass	kilogram	kg
Substance amount	mole	mol
Temperature	Kelvin	K
Time	second	s

Derived SI Units and Defined Equivalents

Quantity:	Unit:	Abbreviation:
Force	newton	N (kg-m/s^2)
Frequency	hertz	Hz (s^{-1})
Liquid volume	liter	l (1000 cm^3)
Quantity of heat	calorie	cal (4.1840 J)
Pressure	pascal	Pa (N/m^2)
Length	angstrom	A (10^{-10} m)
Electric charge	coulomb	C (A-s)
Voltage	volt	V (W/A)
Capacitance	farad	F (C/V)
Inductance	henry	H (V-s/A)
Magnetic flux	weber	Wb (V-s)
Magnetic flux density	tesla	T (V-s/m^2)
Electric resistance	ohm	Ω (V/A)
Power	watt	W (J/s)
Work, Energy	joule	J (N-m)

Fundamental Physical Constants

Name:	**Symbol:**	**Value:**
Avogadro's number	N_0	6.02217×10^{23} /mole
Bohr atomic radius	a_0	5.29177×10^{-11} m
Boltzmann's constant	k	1.38062×10^{-23} J/K
Compton electron wavelength	λ_e	2.42631×10^{-12} m
Electron charge	e	1.60219×10^{-19} C
Electron charge/mass ratio	e/m_e	1.75880×10^{-11} C/kg
Electron rest mass	m_e	9.10956×10^{-31} kg
Faraday	F	96,487 C/equivalent
Gas constant	R	8.31434 J/K-mole
Gravitational constant	G	6.6732×10^{-11} N-m^2/kg^2
Neutron rest mass	m_n	1.67492×10^{-27} kg
Permeability constant	μ_0	$4\pi \times 10^{-7}$ H/m
Permittivity of free space	ε_0	8.85419×10^{-12} F/m
Planck's constant	h	6.6262×10^{-34} J-sec
Proton rest mass	m_p	1.67261×10^{-27} kg
Quantum/charge ratio	h/e	4.13571×10^{-15} J-sec/C
Rydberg constant	R_∞	1.09737×10^{7} /m
Speed of light	c	2.99792×10^{8} m/sec^2
Stefan-Boltzmann constant	σ	5.66961×10^{-8} W/m^2-K^4

Solar System Constants

The Earth

Mass of the earth	5.976×10^{24} kg
Equatorial radius of the earth	6378 km
Polar radius of the earth	6356 km
Volume of the earth	1.087×10^{21} m^3
Average density of the earth	5222 kg/m^3
Angular velocity of the earth about its axis	7.292×10^{-5} rad/sec
Period of the earth's rotation	86,164 sec
Angular velocity of the earth about the sun	1.991×10^{-7} rad/sec
Acceleration due to gravity	9.80665 m/sec^2
Speed of sound in dry air	331.45 m/sec

The Moon

Mass of the moon	7.3505×10^{22} kg
Average radius of the moon	1738 km
Average earth-to-moon distance	384,398 km
Acceleration due to the moon's gravity	1.62 m/sec^2

The Sun

Mass of the sun	1.97×10^{30} kg
Average radius of the sun	696,000 km
Average earth-to-sun distance	1.49×10^{8} km

The Greek Alphabet

Greek Letter:	Name:
$A\ \alpha$	Alpha
$B\ \beta$	Beta
$\Gamma\ \gamma$	Gamma
$\Delta\ \delta$	Delta
$E\ \varepsilon$	Epsilon
$Z\ \varsigma$	Zeta
$H\ \eta$	Eta
$\Theta\ \theta$	Theta
$I\ \iota$	Iota
$K\ \kappa$	Kappa
$\Lambda\ \lambda$	Lambda
$M\ \mu$	Mu
$N\ \nu$	Nu
$\Xi\ \xi$	Xi
$O\ o$	Omicron
$\Pi\ \pi$	Pi
$P\ \rho$	Rho
$\Sigma\ \sigma$	Sigma
$T\ \tau$	Tau
$\Upsilon\ \upsilon$	Upsilon
$\Phi\ \phi$	Phi
$X\ \chi$	Chi
$\Psi\ \psi$	Psi
$\Omega\ \omega$	Omega

Conversion Factors

For your convenience, the HP-22S has built-in SI/English unit conversions for the following:

- Liters ⟷ Gallons
- °Celsius ⟷ °Fahrenheit
- Kilograms ⟷ Pounds
- Centimeters ⟷ Inches
- Radians ⟷ Degrees

When other conversions are needed, consult the table below.

For:	Multiply:	By:
Kilograms	Tons	907.1847
Meters	Feet	0.3048
Meters	Yards	0.9144
Kilometers	Miles	1.60934
Grams	Ounces	28.34952
Cubic centimeters	Fluid ounces	29.57353
Square centimeters	Square inches	6.4516
Square meters	Square feet	0.092903
Square meters	Square yards	0.836127
Cubic centimeters	Cubic inches	16.38706
Cubic meters	Cubic feet	2.83168×10^{-2}
Cubic meters	Cubic yards	0.764555
Joules	BTU	1054.8
Kilowatts	Horsepower	0.745712
Feet	Miles	5280
Knots	MPH	0.868976
Square feet	Acres	43560

Step-by-Step Solutions for Your HP-22S Calculator

Science Student Applications contains a variety of applications, examples, equation-writing hints, and useful tables to help you solve problems in science more easily.

- **Algebra**
  Logarithms • Complex Numbers • Roots of Polynomials
- **Trigonometry**
  Angular Distance and Speed • Determining Altitude of an Object • Area of a Triangle • Multiple Triangles
- **Linear Algebra**
  Dot Product • Cross Product • Perpendicular Component of a Vector • Determinant of a 3 × 3 Matrix • Simultaneous Equations Solution
- **Calculus**
  Derivatives • Integration • Differentials • Taylor Series
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  Force and Acceleration • Simple Harmonic Montion • The Doppler Effect • Fluid Flow • Electric Potential
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