HP-35 ACCURACY ANALYSIS

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The elementary operations such as add, subtract, multiply, divide, reciprocal, and square root all have an accuracy of $\pm$ 1 count in the 10th digit of the answer. All calculations are performed to 12 digits and rounded to 10 digits. This accuracy specification assumes that the operands used are exact to 10 digits.

If the answer resulting from any of these operations is used in a subsequent operation, then this one count of error can be magnified.

For Example:

<u>Actual Answer</u>:

$9(\sqrt{1.11})^2 = 9.99$

<u>HP-35 Answer</u>:

$\sqrt{1.11} = 1.053565375$

(1.053565375) (1.053565375) = 1.109999999

(This answer is off by one count in the 10th digit)

9(1.109999999) = 9.989999991

(This answer is off by 9 counts in the 10th digit)

In all the remaining functions (the transcendental functions), an error specification is more complex. First, there is considerable prescaling utilizing the elementary operations which results in error propagation before the particular function algorithm is utilized. In the case of the trigonometric functions, degrees are converted to radians and the angle is resolved to the positive unit circle. Second, the function itself can magnify any error produced by the prescaling. The tangent function near 90° has a very steep slope, so any small variation in the input argument results in a large variation in the output answer.

For example: A one count variation in the operand 89.5° produces a 2.3×10^{-6} in the answer of tangent 89.5°.

In some of the transcendental functions, post computations are performed. For example:

$$\sin x = \frac{\tan x}{\sqrt{1 + (\tan x)^2}}$$

This can generate additional error and lack of monotonicity* in the answer, therefore, there is an additional absolute error specification of $\pm 1 \times 10^{-9}$ on trigonometric functions. This is an error because they are calculated without small argument resolution. Since Log x is formed by $\frac{\ln x}{\ln 10}$, the Log x function also has an additional error of ± 3 counts in the 10th least significant digit of the displayed answer.

The x^y function, which is found by $e^{y \ln x}$, has one transcendental function followed by another with an intervening multiply. Since each of these operations are internally rounded to 10 digits, the x^y function can show error due to the exponentation of the error produced by the ln and multiply operations. For example: 3^2=9.000000006 (HP-35 answer). A one count error in the ln and multiply operation could yield a 9 count error in the 10th least significant digit of the answer.

The accuracy of the transcendental functions depends on the value of the input argument. Therefore, the accuracy specification is referred to an equivalent error in the input argument.

Trigonometric Accuracy: Answer produced is the exact answer for an input argument somewhere within ± 3 counts in the 10th least significant digit of the input argument. Additionally, there is $\pm 1 \times 10^{-9}$ absolute error in the displayed answer.

* Monotonic function: If any pair of points $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$ then f is an increasing function. Decreasing functions are similarly defined. A function is called monotonic if it is either increasing or decreasing.

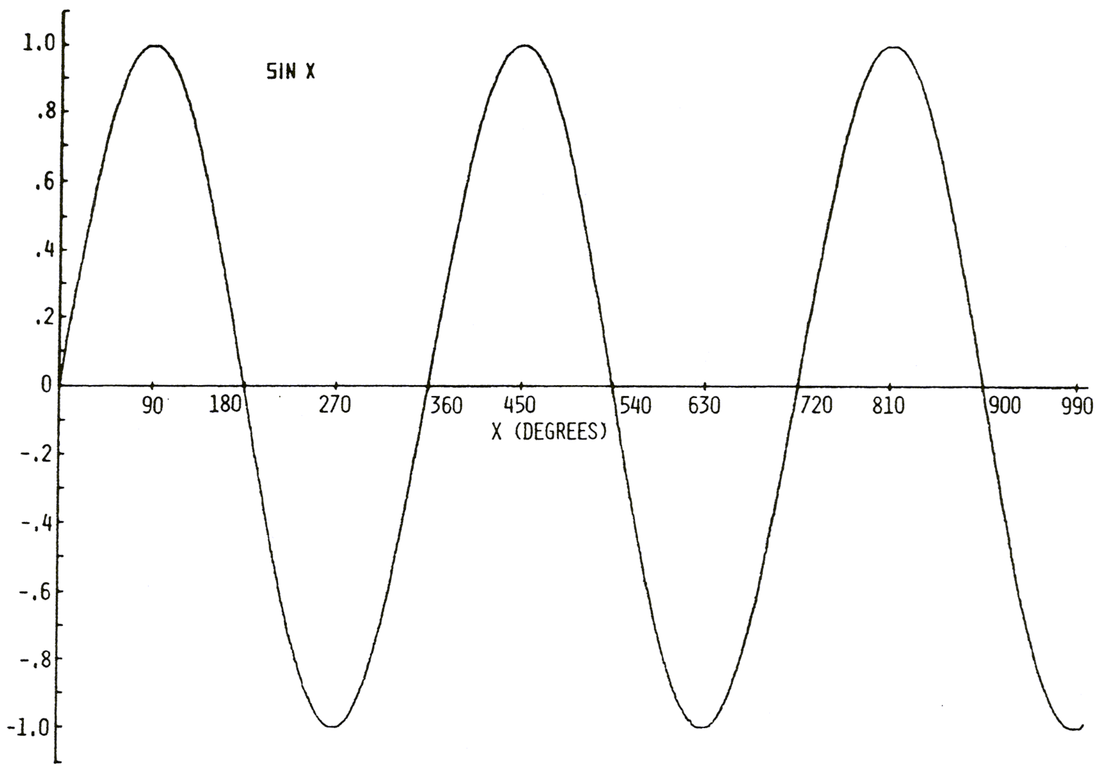
Log x, ln x, and e^x Accuracy: Answer produced is the exact answer for an input argument somewhere within $\pm$ 2 counts in the 10th least significant digit of x. There is an additional error in the ln x of $\pm$ 1 count and in the log x of $\pm$ 3 counts in the 10th least significant digit in the displayed answer.

x^y Accuracy: Answer produced is the exact answer for an input argument somewhere within $\pm$ 4 counts in the 10th least significant digit in x and $\pm$ 10 counts in the 10th least significant digit in y.

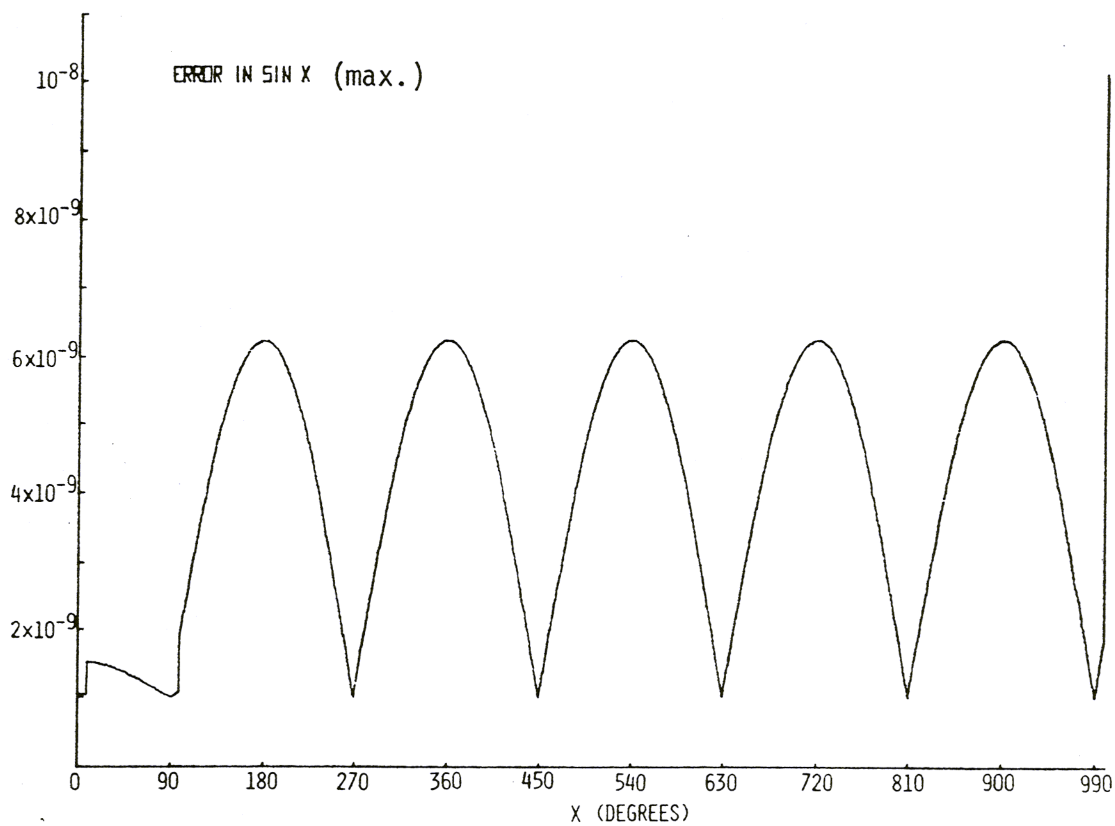
To better show the relationship of maximum error to the various values for x, we have plotted the graphs attached (See Exhibit 1 through 11). The equations used to derive these curves are contained in Exhibit 13.

The peak absolute error in the display for the prescribed range is described in Exhibit 12.

Accuracy statements are given only as a general guide, which to the best of our knowledge, define the maximum error for the respective functions.

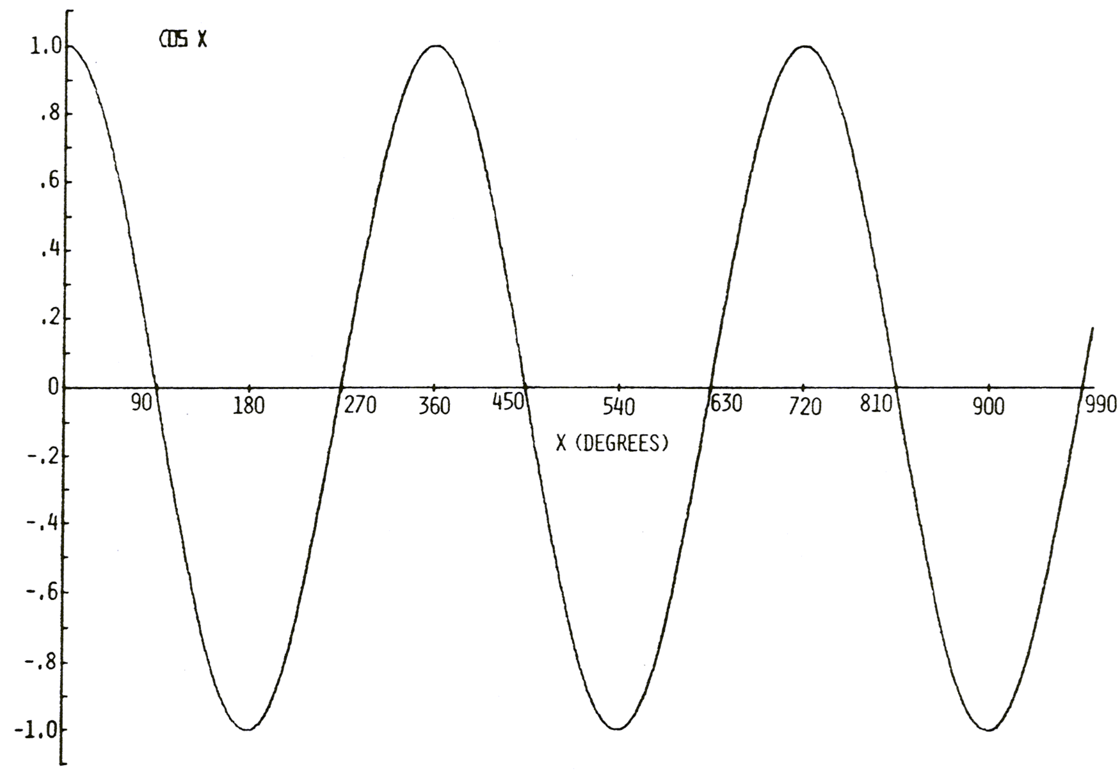


SIN X
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
90
180
270
360
450
540
630
720
810
900
990
X (DEGREES)



ERROR IN SIN X (max.)
10^-8
8x10^-9
6x10^-9
4x10^-9
2x10^-9
0
90
180
270
360
450
540
630
720
810
900
990
X (DEGREES)

EXHIBIT 1


COS X
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
90
180
270
360
450
540
630
720
810
900
990
X (DEGREES)

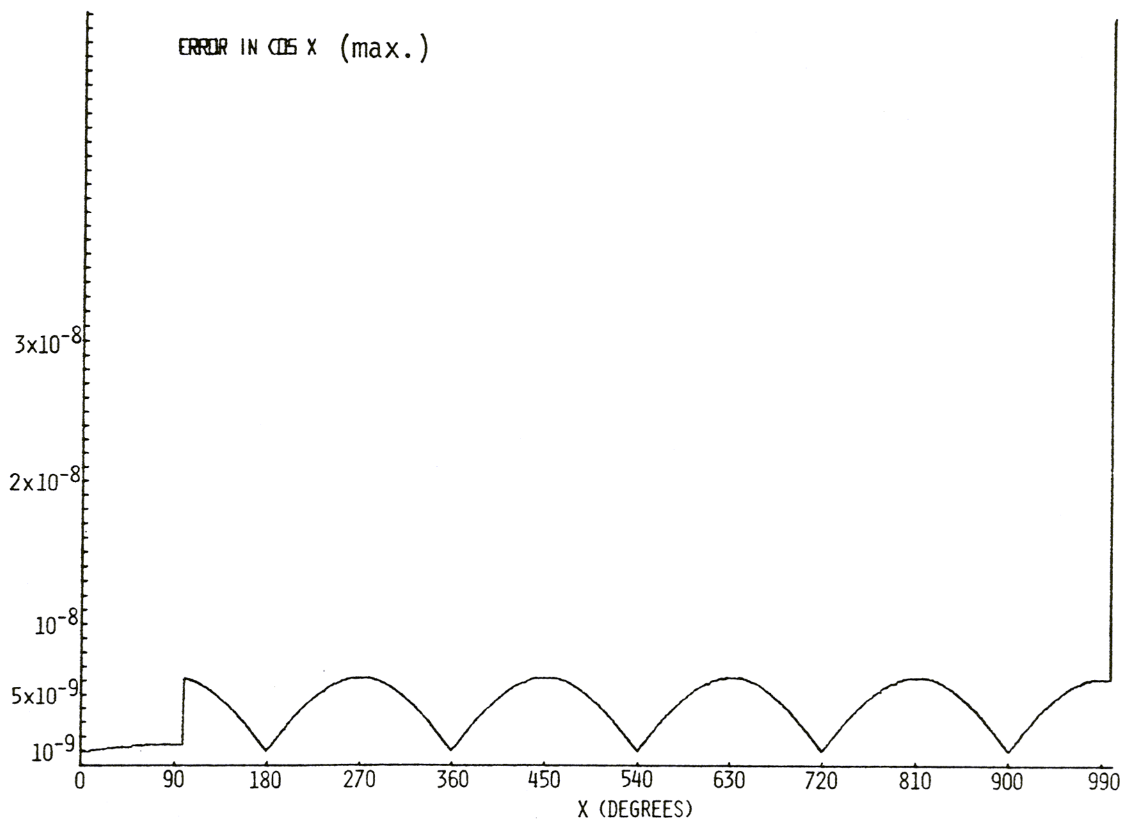
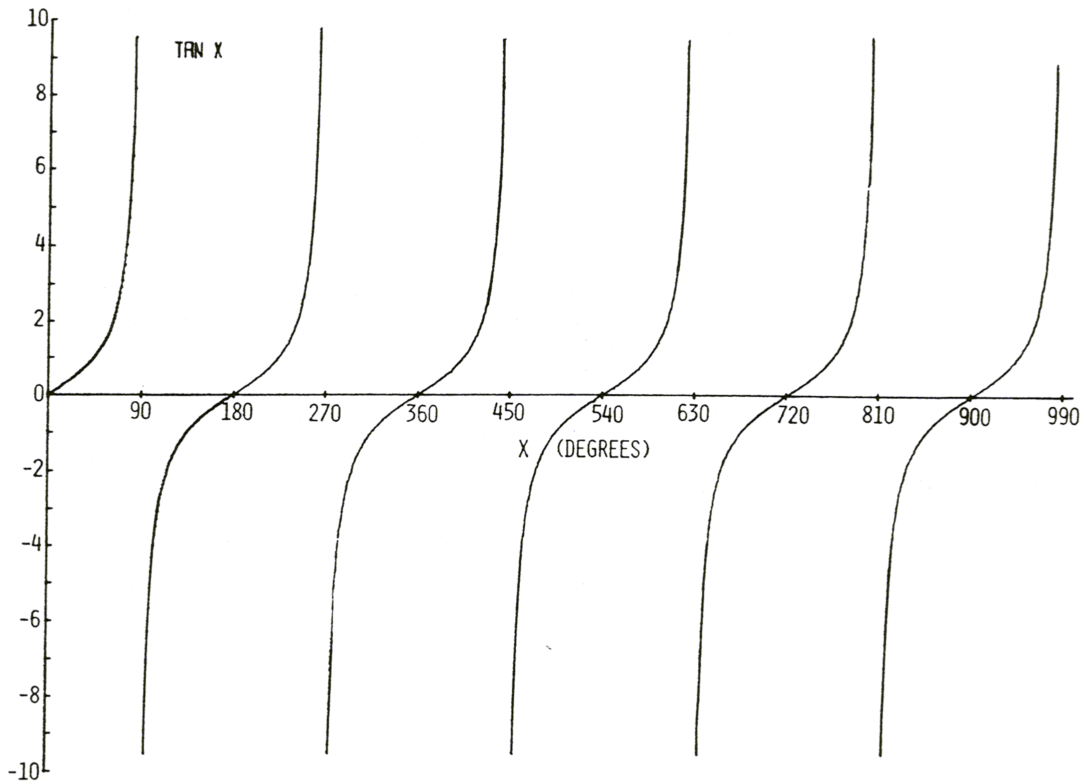

ERROR IN COS X (max.)
3x10^-8
2x10^-8
10^-8
5x10^-9
10^-9
0
90
180
270
360
450
540
630
720
810
900
990
X (DEGREES)

EXHIBIT 2


TAN X
10
8
6
4
2
0
-2
-4
-6
-8
-10
90
180
270
360
450
540
630
720
810
900
990
X (DEGREES)

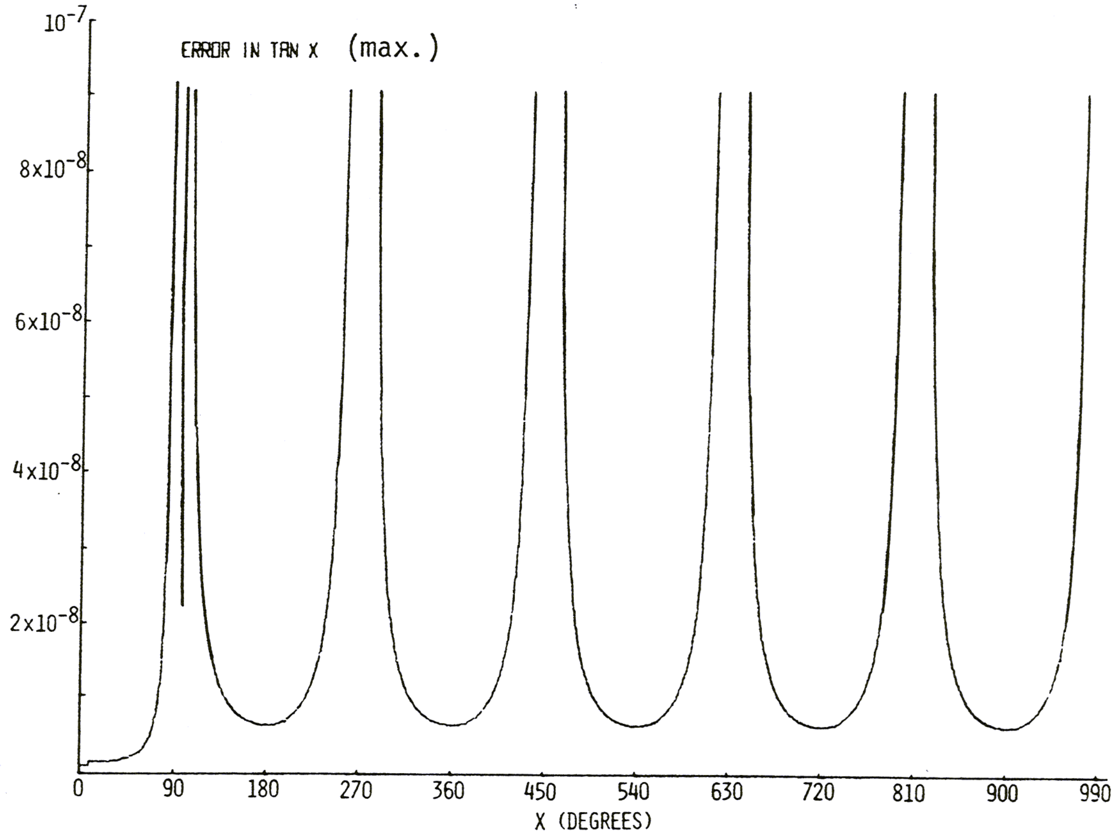
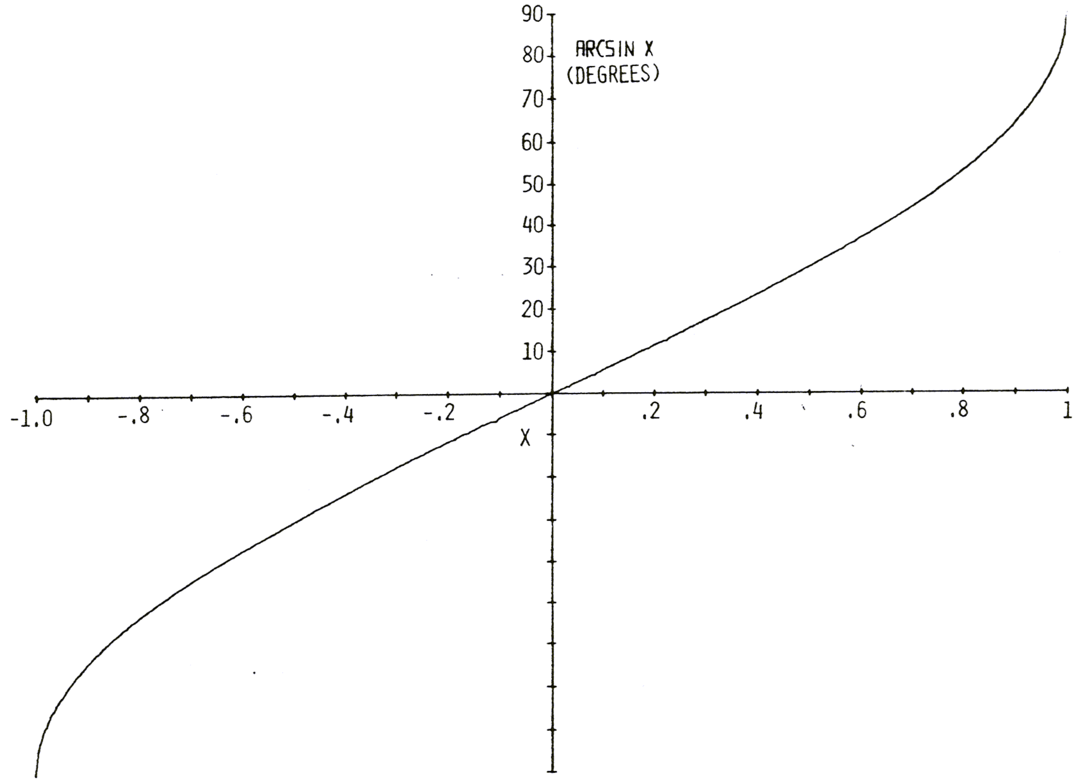
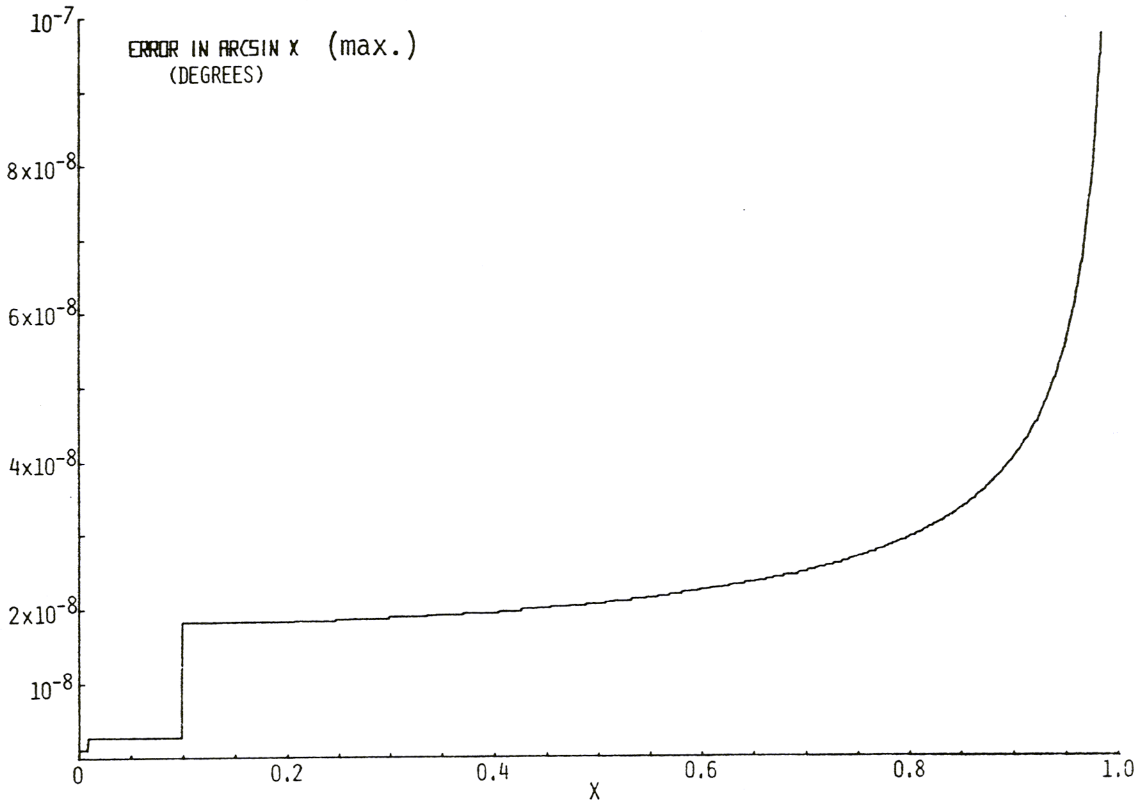

ERROR IN TAN X (max.)
10⁻⁷
8x10⁻⁸
6x10⁻⁸
4x10⁻⁸
2x10⁻⁸
0
90
180
270
360
450
540
630
720
810
900
990
X (DEGREES)

EXHIBIT 3

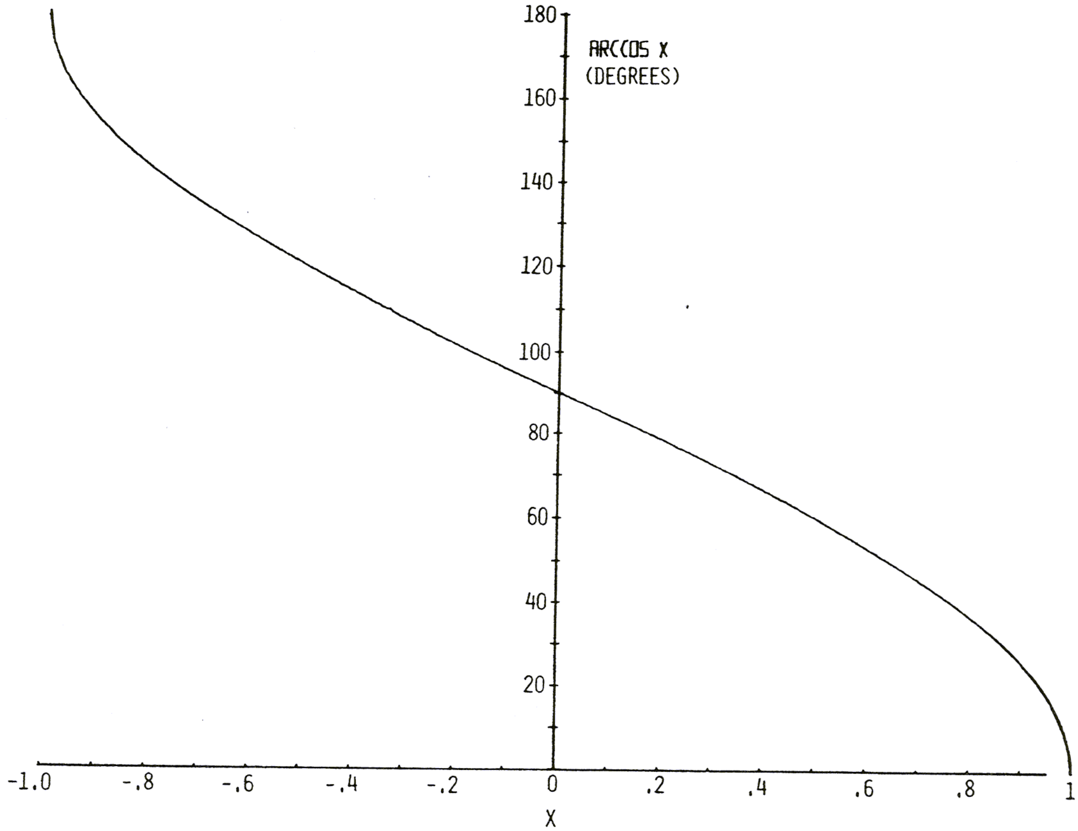


ARCSIN X
(DEGREES)
90
80
70
60
50
40
30
20
10
-1.0
-.8
-.6
-.4
-.2
.2
.4
.6
.8
1
X

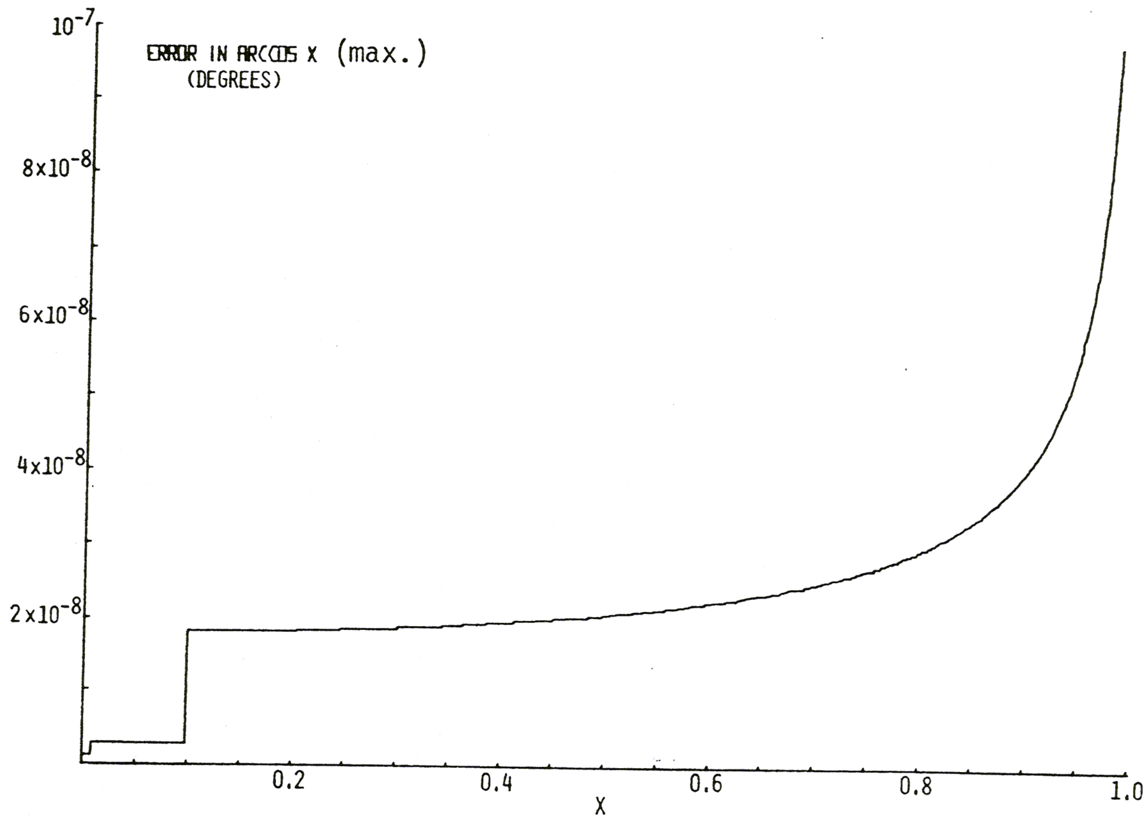


ERROR IN ARCSIN X (max.)
(DEGREES)
10^{-7}
8×10^{-8}
6×10^{-8}
4×10^{-8}
2×10^{-8}
10^{-8}
0
0.2
0.4
0.6
0.8
1.0
X

EXHIBIT 4

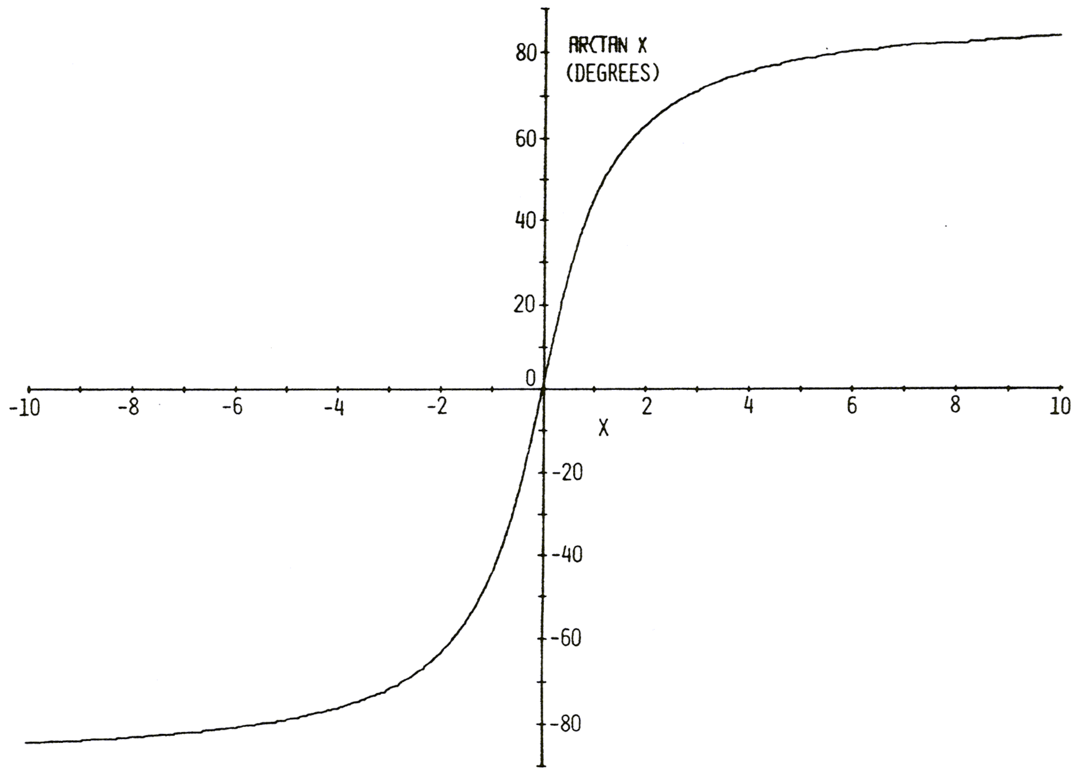


ARCCOS X
(DEGREES)
180
160
140
120
100
80
60
40
20
-1.0
-.8
-.6
-.4
-.2
0
.2
.4
.6
.8
1
X



ERROR IN ARCCOS X (max.)
(DEGREES)
10^-7
8x10^-8
6x10^-8
4x10^-8
2x10^-8
0.2
0.4
0.6
0.8
1.0
X

EXHIBIT 5


ARCTAN X
(DEGREES)
80
60
40
20
0
-20
-40
-60
-80
-10
-8
-6
-4
-2
2
4
6
8
10
X

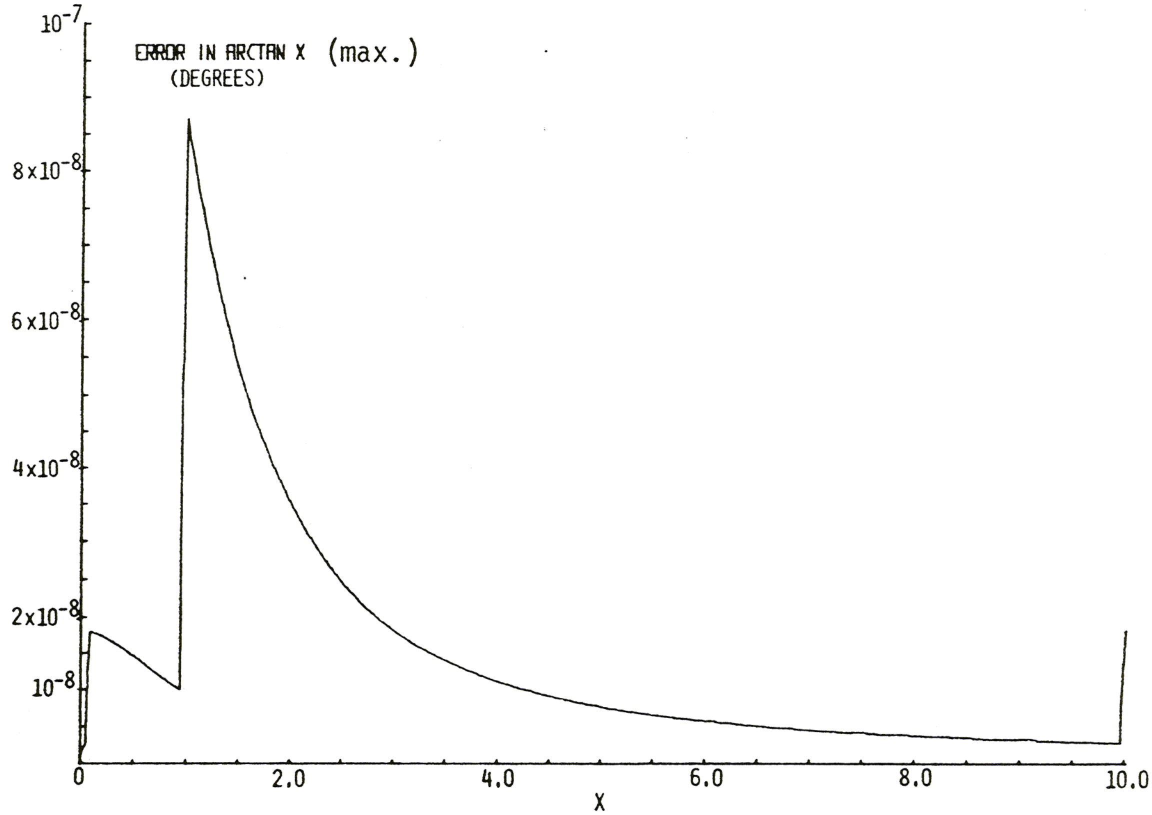
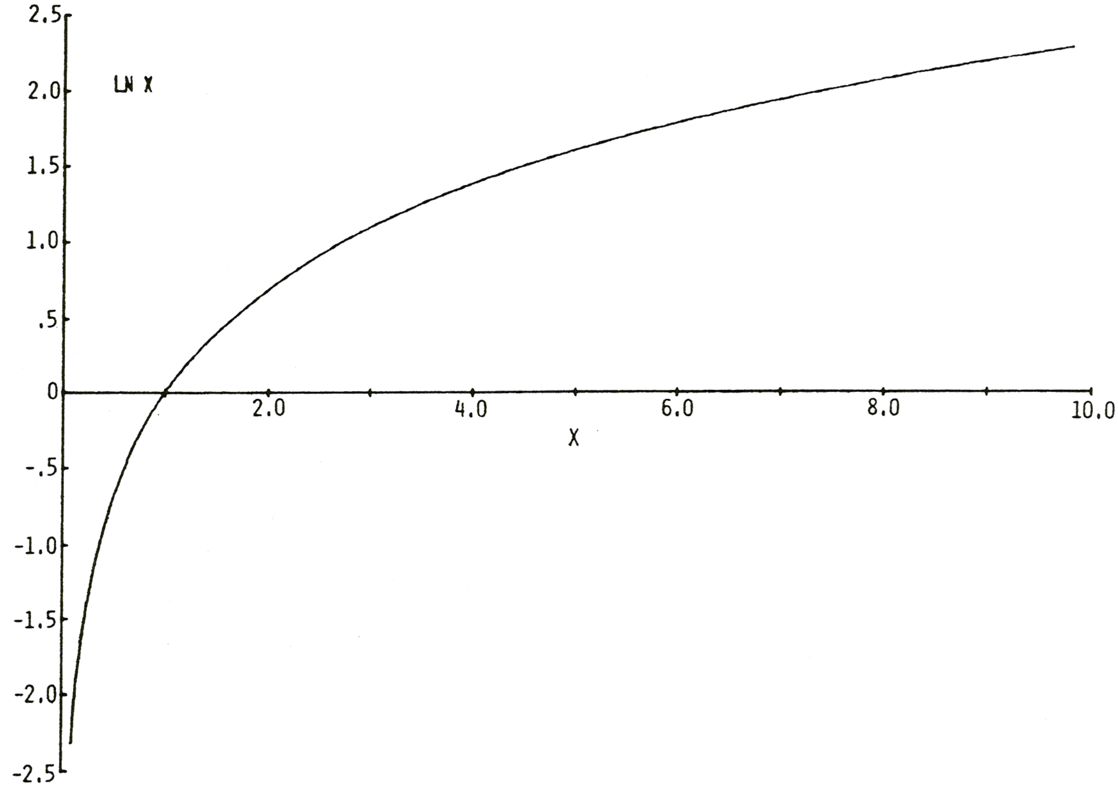

ERROR IN ARCTAN X (max.)
(DEGREES)
10^{-7}
8×10^{-8}
6×10^{-8}
4×10^{-8}
2×10^{-8}
10^{-8}
0
2.0
4.0
6.0
8.0
10.0
X

EXHIBIT 6


2.5
2.0
1.5
1.0
.5
0
-.5
-1.0
-1.5
-2.0
-2.5
LN X
2.0
4.0
6.0
8.0
10.0
X

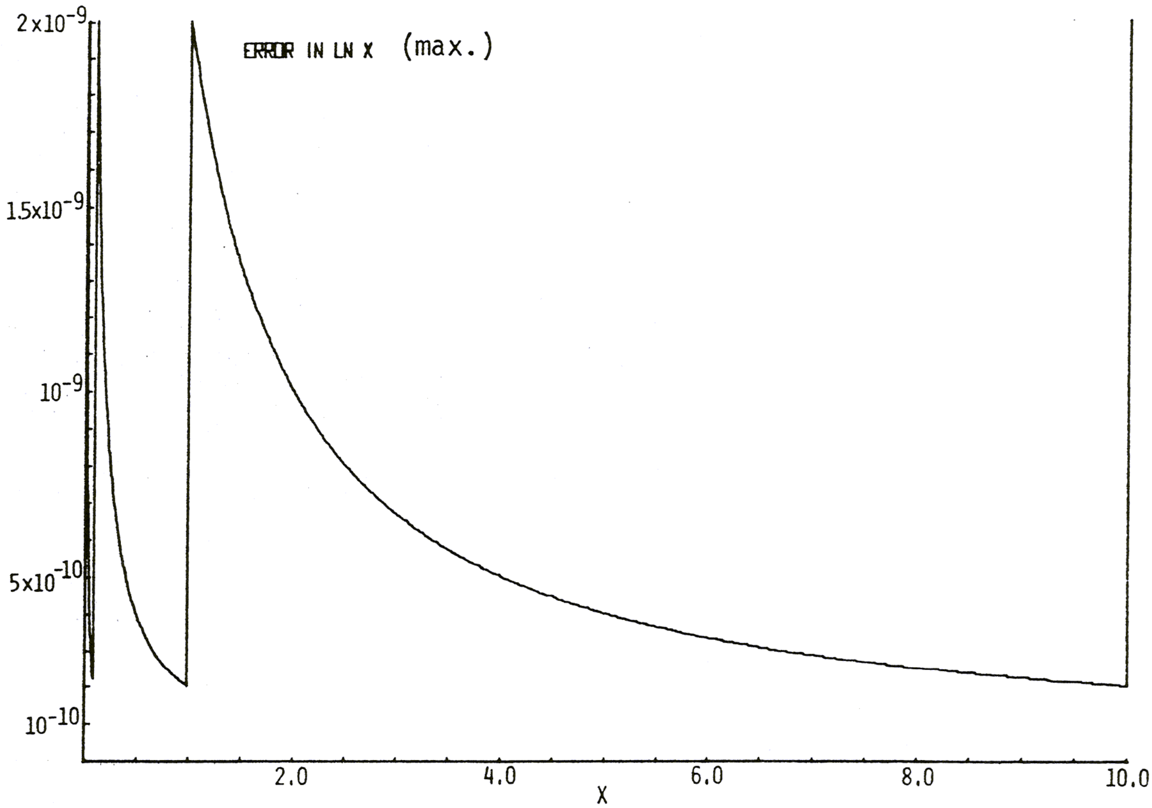
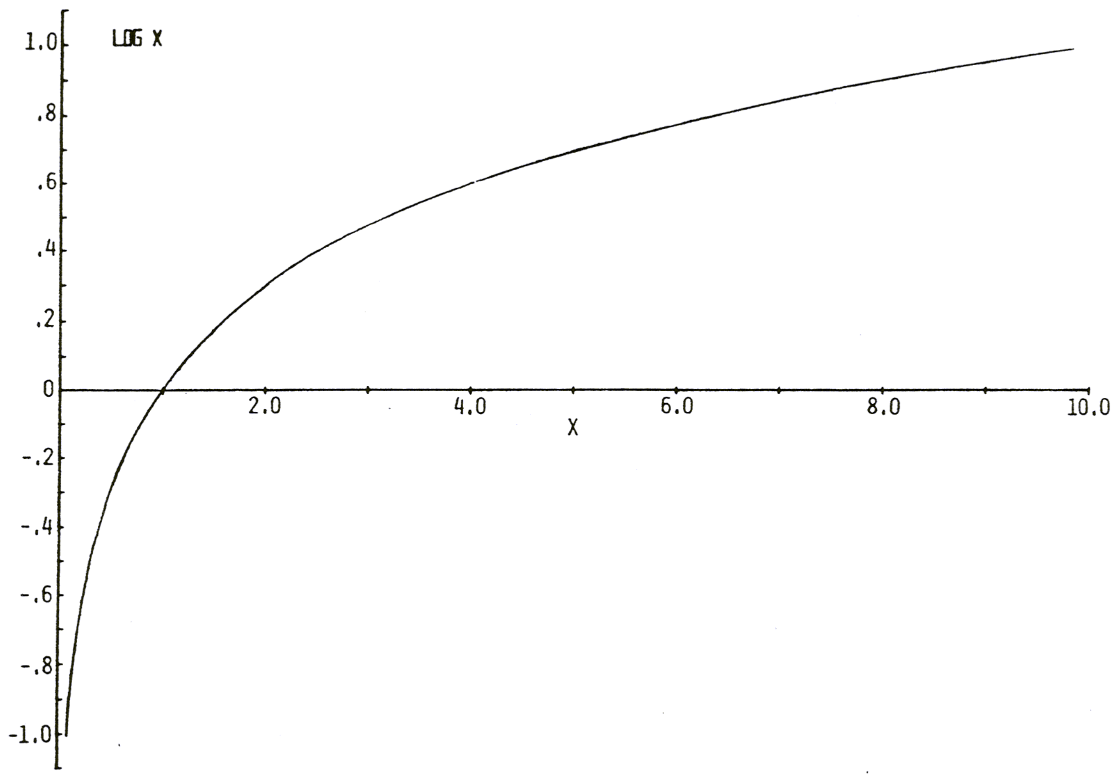
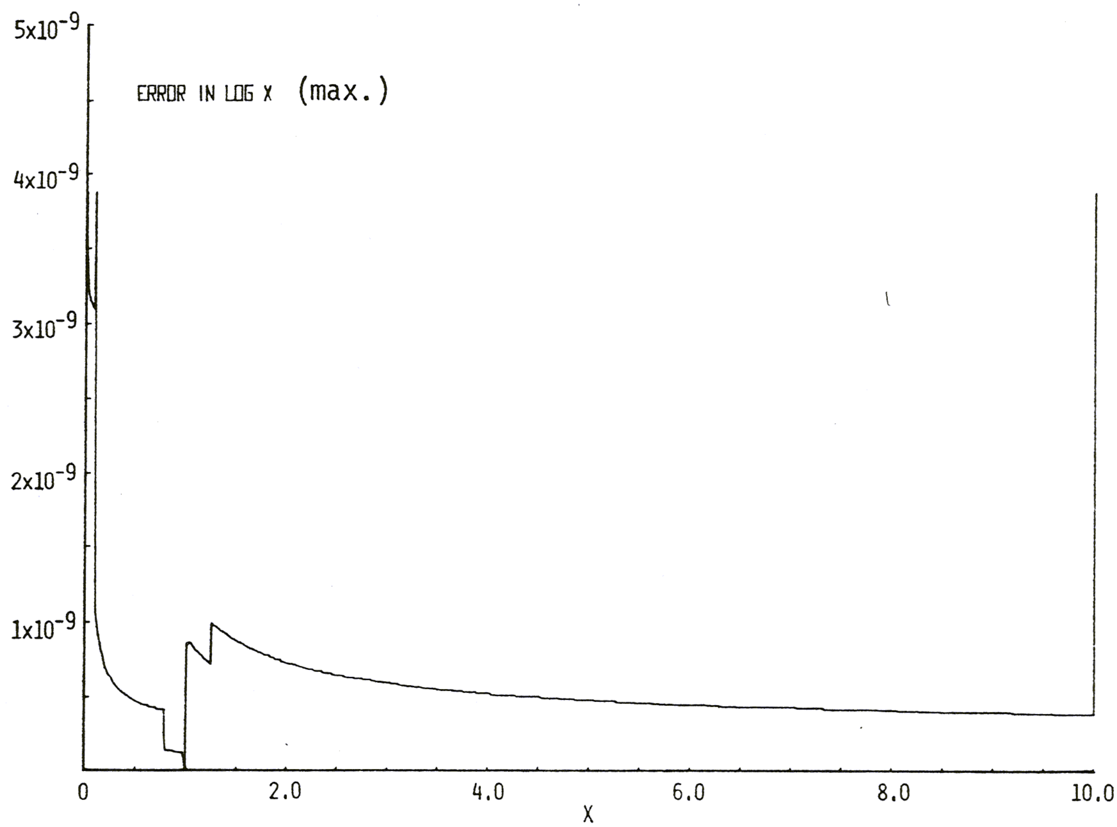

2x10^-9
1.5x10^-9
10^-9
5x10^-10
10^-10
ERROR IN LN X (max.)
2.0
4.0
6.0
8.0
10.0
X

EXHIBIT 7

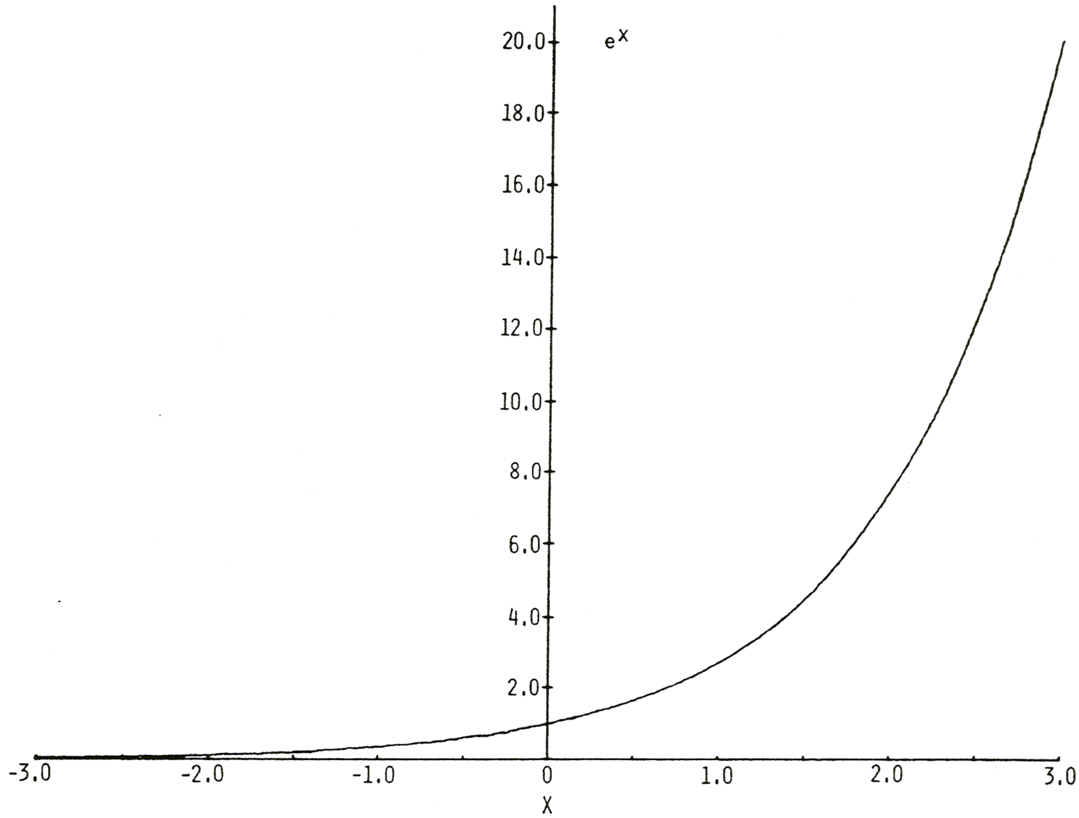


LOG X
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
2.0
4.0
6.0
8.0
10.0
X

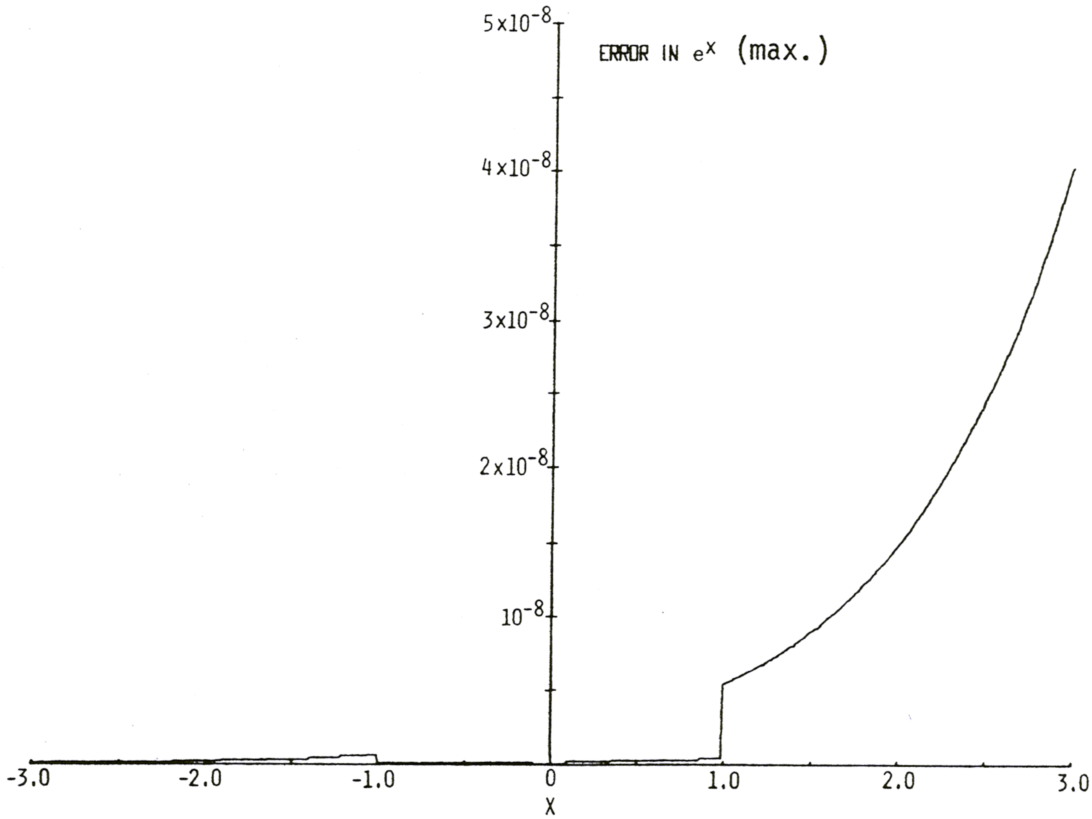


ERROR IN LOG X (max.)
$5x10^{-9}$
$4x10^{-9}$
$3x10^{-9}$
$2x10^{-9}$
$1x10^{-9}$
0
2.0
4.0
6.0
8.0
10.0
X

EXHIBIT 8

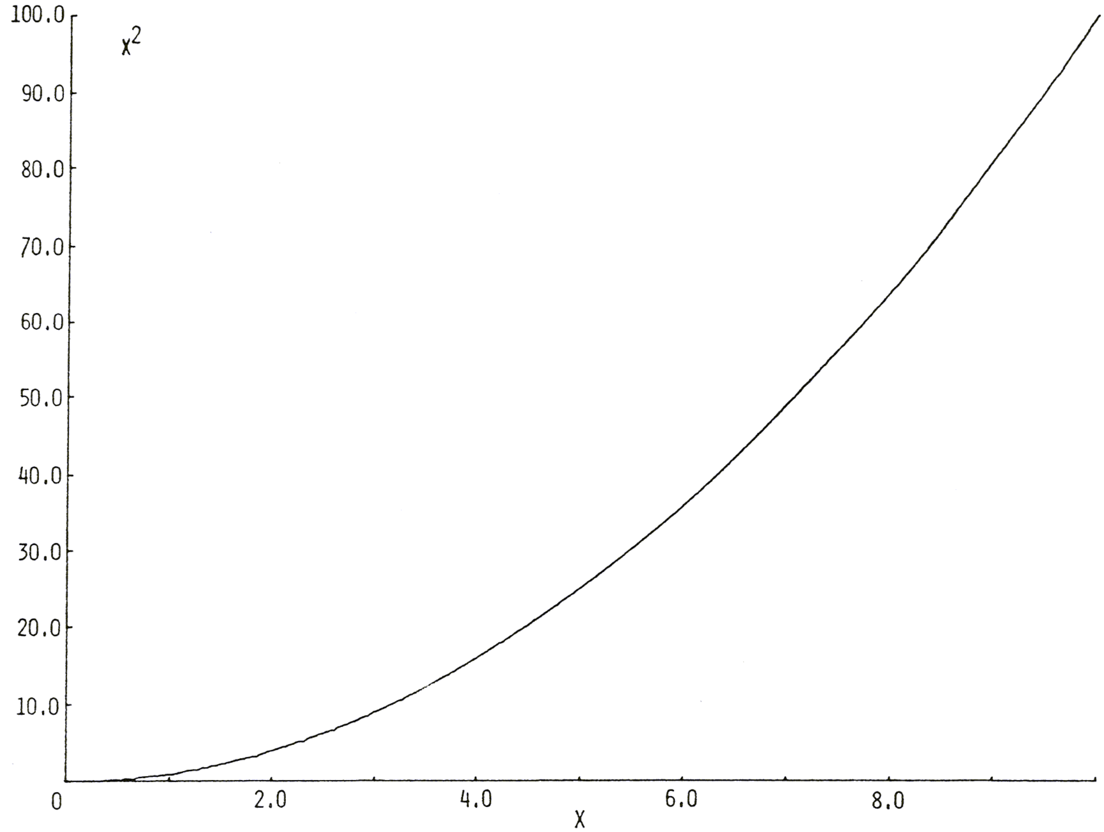


20.0
18.0
16.0
14.0
12.0
10.0
8.0
6.0
4.0
2.0
e^x
-3.0
-2.0
-1.0
0
1.0
2.0
3.0
X

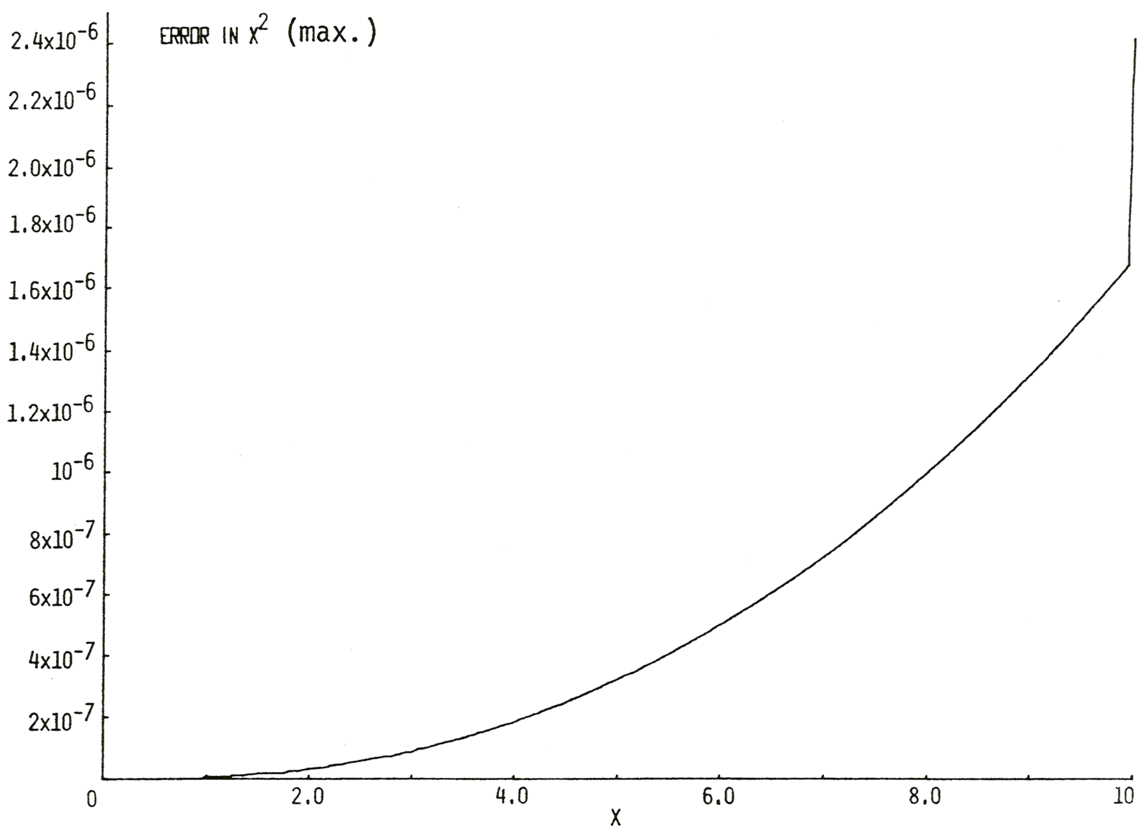


5x10^-8
4x10^-8
3x10^-8
2x10^-8
10^-8
ERROR IN e^x (max.)
-3.0
-2.0
-1.0
0
1.0
2.0
3.0
X

EXHIBIT 9

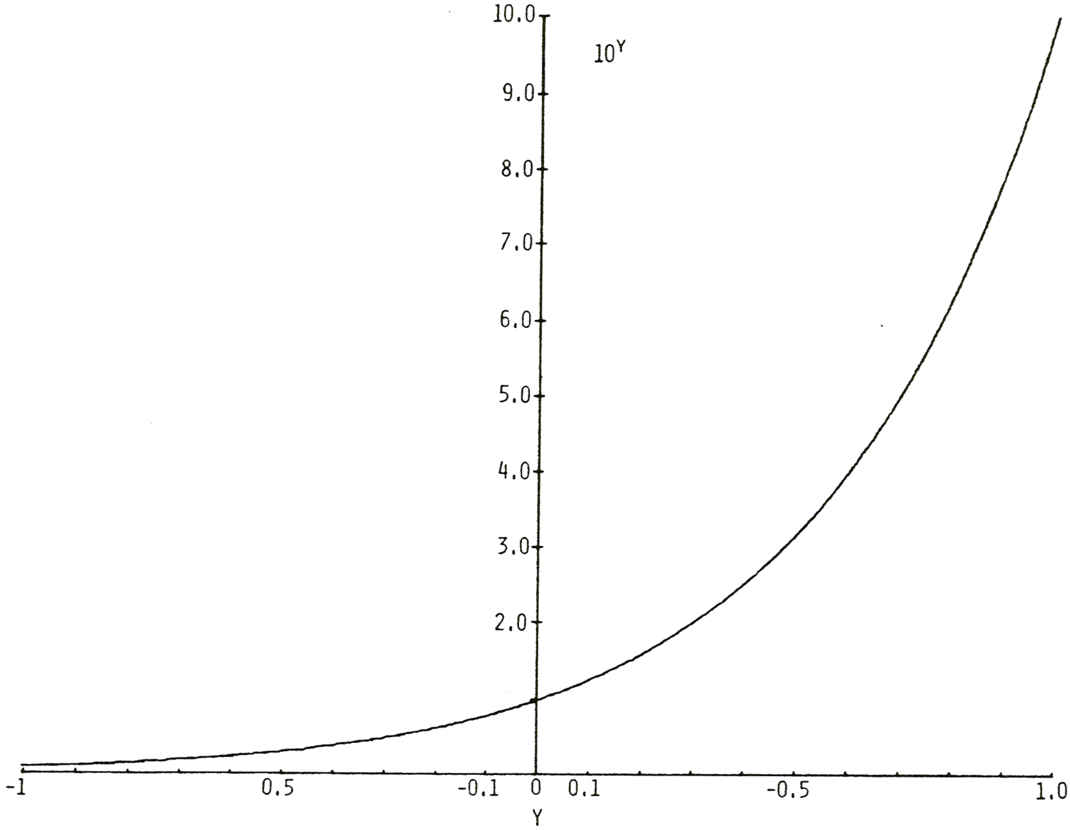


100.0
90.0
80.0
70.0
60.0
50.0
40.0
30.0
20.0
10.0
X2
0
2.0
4.0
6.0
8.0
X

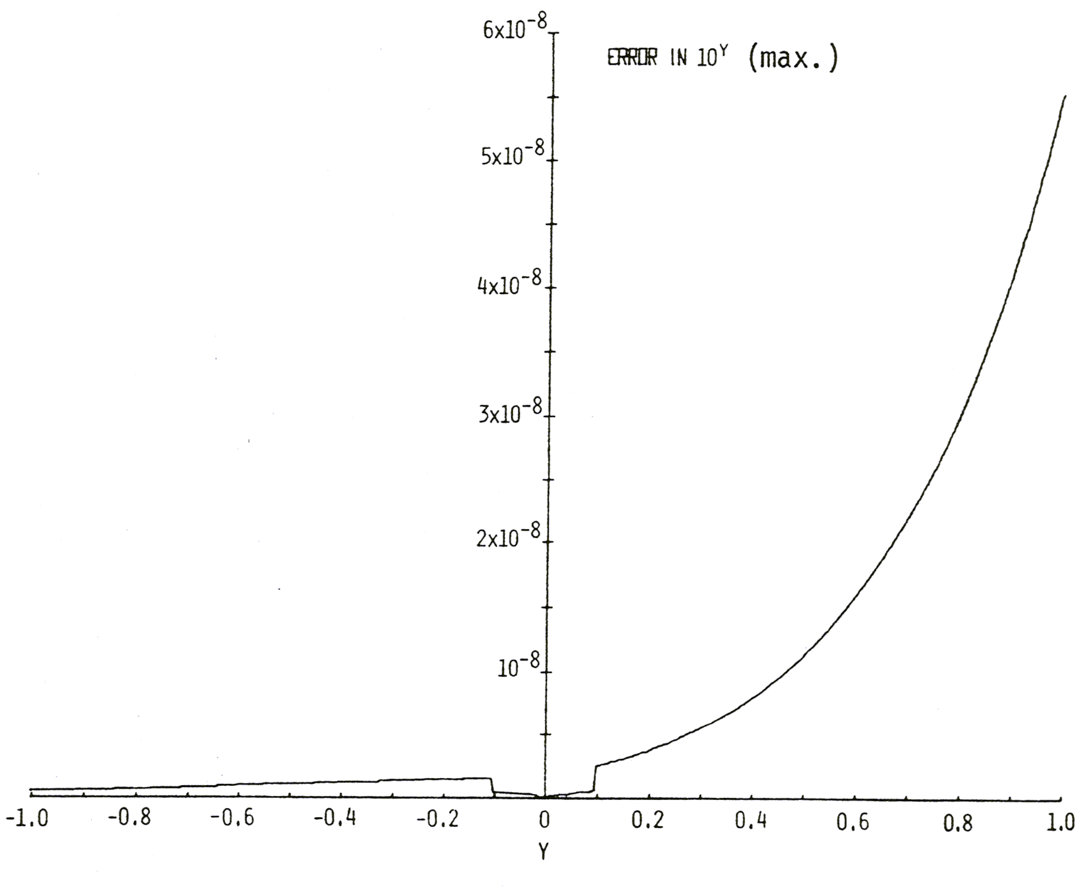


ERROR IN X2 (max.)
2.4x10-6
2.2x10-6
2.0x10-6
1.8x10-6
1.6x10-6
1.4x10-6
1.2x10-6
10-6
8x10-7
6x10-7
4x10-7
2x10-7
0
2.0
4.0
6.0
8.0
10
X

EXHIBIT 10



10.0
10^Y
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
-1
0.5
-0.1
0
0.1
-0.5
1.0
Y



6x10^-8
ERROR IN 10^Y (max.)
5x10^-8
4x10^-8
3x10^-8
2x10^-8
10^-8
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
Y

EXHIBIT 11

Peak absolute error in answer for prescribed ranges

	RANGE	PEAK ABSOLUTE ERROR WITHIN SPECIFIED RANGE
Error in ($\sin x$):	$-900° \leq x \leq 900°$	$E\ max = 6.24 \times 10^{-9}$
Error in ($\cos x$):	$-900° \leq x \leq 900°$	$E\ max = 6.24 \times 10^{-9}$
Error in ($\tan x$):	$-89° \leq x \leq 89°$	$E\ max = 1.72 \times 10^{-6}$
Error in ($\sin^{-1} x$)	$-.99 \leq x \leq .99$	$E\ max = 1.23 \times 10^{-7}$
Error in ($\cos^{-1} x$):	$-.99 \leq x \leq .99$	$E\ max = 1.23 \times 10^{-7}$
Error in ($\tan^{-1} x$):	$-\infty \leq x \leq \infty$	$E\ max = 8.70 \times 10^{-8}$
Error in ($\ln x$):	$.1 \leq x \leq 10$	$E\ max = 2.00 \times 10^{-9}$
Error in ($\log x$):	$.1 \leq x \leq 10$	$E\ max = 3.87 \times 10^{-9}$
Error in (e^x):	$-\infty \leq x \leq 3$	$E\ max = 4.02 \times 10^{-8}$
Error in (x^y):	$1 \leq x \leq 10,\ 0 \leq y \leq 3$	$E\ max = 2.81 \times 10^{-5}$

EXHIBIT 12

Equations for Maximum Error of Function versus Input Argument

$$\text{Error in } (\sin x) = |\cos x| \frac{3}{57.3} \text{ (power of 10 of Least Significant Digit in x)} + 1\times10^{-9}$$

$$\text{Error in } (\cos x) = |\sin x| \frac{3}{57.3} \text{ (power of 10 of Least Significant Digit in x)} + 1\times10^{-9}$$

$$\text{Error in } (\tan x) = (1 + \tan^2 x) \frac{3}{57.3} \text{ (power of 10 of Least Significant Digit in x)} + 1\times10^{-9}$$

$$\text{Error in } (\sin^{-1} x) = \frac{3(57.3)}{\sqrt{1 - x^2}} \text{ (power of 10 of Least Significant Digit in x)} + 1\times10^{-9}$$

$$\text{Error in } (\cos^{-1} x) = \frac{3(57.3)}{\sqrt{1 - x^2}} \text{ (power of 10 of Least Significant Digit in x)} + 1\times10^{-9}$$

$$\text{Error in } (\tan^{-1} x) = \frac{3(57.3)}{1 + x^2} \text{ (power of 10 of Least Significant Digit in x)} + 1\times10^{-9}$$

$$\text{Error in } (\ln x) = \frac{2}{x} \text{ (power of 10 of Least Significant Digit in x)} + \text{(power of 10 of Least Significant Digit in ln x)}$$

$$\text{Error in } (\log x) = \frac{2(.4343)}{x} \text{ (power of 10 of Least Significant Digit in x)} + 3 \text{ (power of 10 of Least Significant Digit in log x)}$$

$$\text{Error in } (e^x) = 2(e^x) \text{ (power of 10 of Least Significant Digit in x)}$$

$$\text{Error in } (x^y) = x^y \left(4\left|\frac{y}{x}\right| \text{(power of 10 of Least Significant Digit in x)} + 10\,|\ln x| \text{ (power of 10 of Least Significant Digit in y)}\right)$$

EXHIBIT 13