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BESSEL FUNCTIONS, ERROR FUNCTION

The first routine computes the Bessel functions $J_n(x)$ and $I_n(x)$, where n is a positive integer and $x > \emptyset$. The second of the two routines finds the error function and complementary error function for positive arguments.

Bessel Functions

The Bessel functions $J_n(x)$ and $I_n(x)$ are computed by generating trial values T_k through the use of recurrence relations. The recurrence is begun at an index m given by

$$m = 2 \text{ INT} \left[\frac{6 + \max(n,z) + \frac{9z}{z+2}}{2} \right]$$

where

$$z = \frac{3x}{2}$$

The initial values selected for recurrence are $T_{m+1} = 10^{-9}$, $T_{m+2} = 0$.

For the functions $J_n(x)$, each term T_k $(0 \leq k \leq m)$ is computed by the relation

$$T_k(x) = \frac{2\ (k+1)}{x} T_{k+1}(x) - T_{k+2}(x)$$

beginning with k = m.

$J_n(x)$ is then found by dividing the term $T_n(x)$ by the normalizing constant

$$K = T_o(x) + 2 \sum_{k=1}^{m/2} T_{2k}(x).$$

After calculating a $J_n(x)$, the values of $J_o(x)$ and $J_1(x)$ may also be found with very little additional computation.

For the functions $I_n(x)$, each T_k is calculated from the recurrence relation

$$T_k(x) = \frac{2\ (k+1)}{x} T_{k+1}(x) + T_{k+2}(x),$$

$0 \leq k \leq m$, beginning with k = m.

$I_n(x)$ is then found from the equation:

$$I_n(x) = e^x \frac{T_n(x)}{T_0(x) + 2\sum_{k=1}^{m} T_k(x)}$$

Error Function

The error function is defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$$

and the complementary error function as

$$\mathrm{erfc}\ (x) = 1 - \mathrm{erf}\ (x).$$

For large values of x (≥ 3), the error function is very close to 1. If erfc(x) is computed as 1 - erf(x), most of the significant figures of erfc(x) will be lost for $x > 3$. Hence two different algorithms are employed in this program, one for $x \leq 3$ and one for $x > 3$. For ≤ 3, the error function is computed by a series sum

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{n=0}^{\infty} \frac{2^n}{1 \cdot 3 \ldots (2n+1)} x^{2n+1}$$

and the complementary error function

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x).$$

For $x > 3$, the complementary error function is computed first, by the asymptotic expansion

$$\mathrm{erfc}(x) = \frac{1}{x\sqrt{\pi}} e^{-x^2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot \ldots \cdot (2n-1)}{(2x^2)^n} \right]$$

and the error function by

$$\mathrm{erf}(x) = 1 - \mathrm{erfc}(x).$$

The accuracy of the calculation of erf(x) and erfc(x) from series sums may be controlled by the user. For $x \leq 3$, it is quite reasonable to specify 9 for maximum accuracy; for $x > 3$, the series may never converge with 9, and a safer specification would be 6 digits.

Remarks:

1. The range of values $0 \leq x \leq 10^{-6}$ is out of bounds for the Bessel functions in this program. In this range, however, one may take $J_0(x) = J_0(0) = I_0(x) = I_0(0) = 1$, and $J_n(x) = J_n(0) = I_n(x) = I_n(0) = 0$, $n \neq 0$.

2. The computation of erfc(x) will halt on overflow for $x \geqslant 15$.

Reference: Abramowitz and Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, 1968.

Examples:

1. Find J_5 (9.2)

2. Find J_0 (9.2)

3. Find J_1 (9.2)

4. Find I_3 (4.7)

5. Find erf and erfc 4.55 to 6 places

	Keystrokes:	Display:	
	[USER]		(set USER mode)
	[XEQ] [ALPHA] SIZE [ALPHA] 007		
	[XEQ] [ALPHA] INIT [ALPHA]		
	[J]	N?	
	5 [R/S]	X?	
1)	9.2 [R/S]	J=-0.1005	
2)	[R/S]	J0=-0.1367	
3)	[R/S]	J1=0.2174	
	[I]	N?	
	3 [R/S]	X?	
4)	4.7 [R/S]	I=7.4195	
	[E]	ACCURACY?	
	6 [R/S]	X?	
5)	4.55 [R/S]	ERF=1.000000	
	[R/S]	ERFC=1.237405E-10	

User Instructions

SIZE: 007

STEP	INSTRUCTIONS	INPUT	FUNCTION	DISPLAY
1	Load program and set USER mode		[USER]	
2	Initialize		[XEQ] INIT	
3	To find $J_n(x)$		[J]	N?
	input n	n	[R/S]	X?
	input x	x	[R/S]	J=
	(optional) to find $J_0(x)$		[R/S]	J0=
	and $J_1(x)$		[R/S]	J1=
4	To find $I_n(x)$		[I]	N?
	input n	n	[R/S]	X?
	input x	x	[R/S]	I=
5	To find error		[E]	ACCURACY?
	input no. of decimal places	n	[R/S]	X?
	input x	x	[R/S]	ERF=
			[R/S]	ERFC=

00060C Program Listings

```
 01*LBL "INI      "J"
T"
 02 CLRG
 03 RTN
 04*LBL J
 05 XEQ a
 06 SF 00
 07*LBL 09
 08 XEQ b
 09 CF 02
 10 ST+ 00
 11 XEQ b
 12 FS?C 02
 13 GTO 09
 14 RCL 03
 15 RCL 00
 16 ENTER↑
 17 +
 18 RCL 05
 19 -
 20 /
 21 "J="
 22 XEQ d          - - - - - - - - -
 23 GTO C          input n + x
 24*LBL a
 25 FIX 4
 26 "N?"
 27 PROMPT
 28 STO 01
 29 "X?"
 30 PROMPT         - - - - - - - - -
 31 1.5            initialization
 32 *              for Bessel
 33 STO 03         (Jn + In)
 34 RCL 01
 35 X<=Y?
 36 X<>Y
 37 6
 38 +
 39 RCL 03
 40 9
 41 *
 42 RCL 03
 43 2
 44 +
 45 /
 46 +
 47 2
 48 /
 49 INT
 50 ENTER↑
 51 +
 52 2
 53 +
 54 STO 06
 55 3
 56 RCL 03
 57 /
 58 STO 02
 59 0
 60 STO 05
 61 STO 00
 62  E-9
 63 STO 04
 64 RTN
 65*LBL b          - - - - - - - - -
 66 DSE 06         Compute one term
 67 SF 02          FZ set except
 68 RCL 06         for F = 0
 69 RCL 01
 70 X≠Y?
 71 GTO 00
 72 RCL 04
 73 STO 03
 74*LBL 00
 75 RDN
 76 RCL 05
 77 FS? 00
 78 CHS
 79 X<>Y
 80 RCL 02
 81 *
 82 RCL 04
 83 STO 05
 84 *
 85 +
 86 STO 04
 87 RTN
 88*LBL C          - - - - - - - - -
 89 RCL 05         Compute J0(x) +
 90 RCL 00         J1(x)
 91 ENTER↑
 92 +
 93 RCL 05
 94 -
 95 /
 96 "J0="
 97 XEQ d
 98 RCL 04
 99 CHS
100 RCL 00
101 ENTER↑
102 +
```

000600

Program Listings

103-203 missing

```
204 X<>Y
205 RND
206 X=Y?
207 GTO 00
208 LASTX
209 GTO 06
210◆LBL 00
211 LASTX
212 RCL 04
213 /
214 1
215 X<>Y
216 -
217 LASTX
218 X<>Y
219◆LBL 02
220 "ERF="
221 XEQ d
222 X<>Y
223 "ERFC="
224◆LBL d
225 ARCL X
226 AVIEW
227 STOP
228 .END.
```

- - - - - - - - -
erfc(x)

- - - - - - - - -
erf(x)

- - - - - - - - -
display

000600

REGISTERS, STATUS, FLAGS, ASSIGNMENTS

DATA REGISTERS

00	ΣT_k	50	
	n; erf term		
	2/x		
	1.5x, T_n		
	T_k; $(e^{x^2}\sqrt{\pi})^{-1}$		
05	T_{k+1}	55	
	K; places		
10		60	
15		65	
20		70	
25		75	
30		80	
35		85	
40		90	
45		95	

STATUS

SIZE 007 TOT. REG. 53 USER MODE
ENG ___ FIX 09 SCI ___ ON X OFF ___
DEG ___ RAD ___ GRAD ___

FLAGS

#	INIT S/C	SET INDICATES	CLEAR INDICATES

ASSIGNMENTS

FUNCTION	KEY	FUNCTION	KEY

BESSEL FUNCTIONS
ERROR FUNCTION
PROGRAM REGISTERS NEEDED: 47

ROW 1 (1 - 5)

ROW 2 (5 - 11)

ROW 3 (11 - 20)

ROW 4 (21 - 25)

ROW 5 (26 - 32)

ROW 6 (33 - 45)

ROW 7 (46 - 58)

ROW 8 (59 - 67)

ROW 9 (67 - 77)

ROW 10 (78 - 89)

ROW 11 (90 - 97)

ROW 12 (98 - 107)

ROW 13 (107 - 113)

ROW 14 (113 - 122)

ROW 15 (123 - 130)

ROW 16 (131 - 133)

ROW 17 (134 - 144)

ROW 18 (145 - 156)

ROW 19 (157 - 169)

ROW 20 (170 - 180)

ROW 21 (181 - 192)

ROW 22 (193 - 205)

ROW 23 (206 - 216)

ROW 24 (217 - 223)

ROW 25 (223 - 228)

ROW 26 (228 - 228)