00609C PROGRAM DESCRIPTION I

Program Title Fitting Polynomials of Degree M to Data

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Program Description, Equations, Variables The problem is the following. Given n data pairs (x_i, y_i), i=1,2,...,n, find the polynomial of degree m, $Q_m(x)$, which minimizes

$$S = \sum(y_i - Q_m(x_i))^2,$$

where the sum (and all following sums) extends from i=1 to n. Rather than solving the system of simultaneous linear equations derived by setting the partial derivatives of S with respect to each of the coefficients of Q_m equal to zero, this program utilizes another method which appears to avoid the problem of loss of accuracy due to the matrix inversion involved in the usual method. We begin by constructing m+1 polynomials, $P_j(x)$, j=0,1,...,m, with the property: $S = \sum P_j(x_i)P_k(x_i) = 0$ if $j \neq k$.

The desired polynomial $Q_m(x)$ is then easily expressed as a linear combination of these basis polynomials if their form is chosen properly. These polynomials are constructed as follows: $P_0(x) = 1$, $P_1(x) = x + B_1$, and
$P_j(x) = (x + B_j)P_{j-1}(x) + C_j P_{j-2}(x)$, for j=2,3,...,m.

Necessary Accessories number of memory modules = 1 +INT((56 +2n)/64)

Operating Limits and Warnings No internal scaling is performed. When dealing with very large or very small numbers for x and/or y, scale the numbers before enterring them.

Reference(s) This was adapted from an old (1966) FORTRAN program written for the Ames Laboratory of the U.S.A.E.C. (program distribution #360400266110003).

00609C PROGRAM DESCRIPTION I (continued)

The requirement that $P_j(x)$ be orthogonal to $P_{j-1}(x)$ and $P_{j-2}(x)$ yields

$$B_j = -\frac{\sum x_i P_{j-1}^2(x_i)}{\sum P_{j-1}^2(x_i)}, \qquad C_j = -\frac{\sum P_{j-1}^2(x_i)}{\sum P_{j-2}^2(x_i)}.$$

The orthogonality of $P_j(x)$ to lower order polynomials is assured by the form of the recursion relation.

If we let $Q_m(x) = \sum_{k=0}^{m} D_k P_j(x)$ and solve the m+1 least squares equations, we have,

$$D_k = \frac{\sum y_i P_k(x_i)}{\sum P_k^2(x_i)}.$$

Then the definition of $P_j(x)$ can be used to convert the D_j coefficients to the form $\hat{y}_i = a_0 + a_1 x_i + a_2 x_i^2 + \ldots + a_m x_i^m$.

To test the goodness of the least squares fit, two quantities are computed, the correlation coefficient given by

$$r^2 = \frac{\sum(\hat{y}_i - \bar{y})^2}{\sum(y_i - \bar{y})^2} \qquad \text{where} \quad \bar{y} = \sum y_i/n,$$

and the mean deviation of the computed curve from the data

$$YMD = \sum |\hat{y}_i - y_i|/n.$$

For a good fit r^2 will be close to 1.0 and YMD will be small compared to the values of y_i.

As written the program works for $m \leq 6$. There is no inherent reason why m could not be as large as 10 or 15. This can be done by modifying several lines of the program. If M is the largest value of m which will be used, replace the following numbers as indicated.

old #	new #	in lines
6	M	92 and 128
7	1+M	105, 115, 194
23	17+M	99, 147, 189
24	18+M	68, 156
30	18+2M	87, 125
36	18+3M	297
37	19+3M	3, 16, 301 (37.9 becomes (19+3M).9)

Required number of memory modules = 1 +int((38 +3M +2n)/64).

Note storage begins at Reg 19+3m.

SIZE = 19+3m + 2n

PROGRAM DESCRIPTION II

Sample Problem (Sketch if Desired) An Object that is thrown off of a building speeds up as it falls due to the acceleration of gravity. The position of an object can be represented by: $x = x_o + v_o t + \frac{1}{2}gt^2$,
where x is the position, x_o the initial height, v_o the initial velocity, and g is the acceleration due to earth's gravity. The following data were obtained for a rubber ball.

t (s) = 1 2 3 4 5
x (m) = 153 150 135 95 70

What was the initial height and velocity of the ball?

Where is it at 6 seconds?

What polynomial exactly goes through the above five points?

SOLUTION:

Input	Function	Display	Comments
	(USER)		set USER mode
	(XEQ) SIZE 047		allocate registers (37 +2n)
	CF 29		improves labels on output
	(XEQ) LSQ	X ENT Y, A	prompt for data entry
1	(ENTER↑)	1.000	input the five data pairs
153	(A)	1.000	
2	(ENTER↑)	2.000	
150	(A)	2.000	
3	(ENTER↑)	3.000	
135	(A)	3.000	
4	(ENTER↑)	4.000	
95	(A)	4.000	
5	(ENTER↑)	5.000	
70	(A)	5.000	
1	(B)	A0=1.8690E2	first check to see if a linear
	(R/S)	A1=-2.2100e1	equation fits the data -
	(C)	R↑2=9.1511E-1	not very good fit
	(R/S)	YMD=8.6800E0	
2	(b)	A0=1.5240E2	x_o fit a quadratic curve
	(R/S)	A1=7.4714E0	v_o
	(R/S)	A2=-4.9286E0	$\frac{1}{2}g$
	(C)	R↑2=9.7882E-1	good fit to the data
	(R/S)	YMD=4.1486E0	
6	(D)	19.800	height in meters at 6 seconds
4	(b)	A0=2.1000E2	coefficients of a curve which
	(R/s)	A1=-1.1925E2	will go through all five points
	(R/S)	A2=8.4292E1	
	(R/S)	A3=-2.4250E1	
	(R/s)	A4=2.2083E0	
	(C)	R↑2=1.0000E0	YMD/150 = 8.7E-9, which is the
	(R/S)	YMD=1.3000E-6	limit set by the 10 digits of
			the calculator.

The (R/S)'s shown above are not necessary when the printer is attached.

00609C

USER INSTRUCTIONS

*PROG WIPES OUT ALL REGISTERS BY CLRG

SIZE: (HP-41C) 37 +2n

STEP	INSTRUCTIONS	INPUT	FUNCTION	DISPLAY
1	Enter the program or load the 6 sides of			
	program cards, and set USER mode.			
2	Execute LSQ.		(XEQ) LSQ	X ENT Y, A
3	Enter (x,y) pairs, i=1,2,...,n. Calcula-	x_i	(ENTER↑)	x_i
	tor will respond with number entered.	y_i	(A)	i
4	Option: Remove a previously enterred	x_i	(ENTER↑)	x_i
	pair. Must be enterred exactly as done	y_i	(a)	n -1
	originally, other wise DATA ERROR will			
	be displayed.			
5	Compute the coefficients for a least	m	(B)	a_0
	squares fit to a polynomial of degree m.		(R/S)	a_1
			⋮ (R/S)	⋮ a_m
6	Compute the correlation coefficient and		(C)	r^2
	mean deviation of curve from data.		(R/S)	YMD
7	For a given x value, compute $\hat{y}$ using	x	(D)	$\hat{y}$
	the fitted polynomial.			
8	Option: Relist the coefficients.		(d)	a_0
			(R/S)	a_1
			⋮ (R/S)	⋮ a_m
9	Option: To list the (x,y) pairs that		(XEQ) L	X1
	were enterred in step 3, execute L.		(R/S)	Y1
			(R/S)	X2
			⋮	⋮
			(R/S)	Ym

SIZE: (HP-41C)

STEP	INSTRUCTIONS	INPUT	FUNCTION	DISPLAY
10	To save execution time when fitting	m	(B)	a_o
	curves of several degrees to one set of		(R/S)	a_1
	data points, compute the curve of lowest		⋮	⋮
	degree with B (step 4). Then with the		(R/S)	a_m
	next higher order execute b. Any of	m'	(b)	a_o
	the steps 6-9 may be done between these		(R/S)	a_1
	least squares fits, but not steps 3 or 4.		⋮	⋮
	If the data is alterred, begin again		(R/S)	$a_{m'}$
	with executing B.			
11	To start over with a new set of data,		(XEQ) LSQ	X ENT Y, A
	execute LSQ and go to step 3.			
	NOTES			
a	When the printer is attached, the (R/S)'s shown above are not necessary.			
b	Execution time is approximately 4nm seconds when executing with B, and			
	4 n(m' -m) with b (step 10).			
c	The program is 570 bytes long. If one is limited in the number of memory			
	modules that are available, the routine for listing the (x,y) data pairs			
	can be deleted (program steps 329 - 360).			

PROGRAM LISTING

☐ 67 ☐ 97 ☒ 41C

STEP/LINE	KEY ENTRY	KEY CODE (67/97 only)	COMMENTS
01	♦LBL "LSQ"		clear registers and initialize loop for enterring (x,y) pairs
02	CLRG		
03	37.9		
04	STO 02		
05	"X ENT Y, A"		
06	AVIEW		prompt user
07	STOP		
08	♦LBL A		enter data
09	X<>Y		
10	STO IND 02		x_i
11	X<>Y		
12	ISG 02		
13	STO IND 02		y_i
14	ISG 02		
15	RCL 02		
16	37		
17	−		compute the number of enterred pairs
18	2		
19	/		
20	INT		
21	STO 04		
22	XEQ 00		set up loop index
23	RCL 04		
24	STOP		
25	GTO A		
26	♦LBL a		remove a pair
27	STO 13		$y_i \rightarrow R_{13}$
28	X<>Y		
29	STO 12		$x_i \rightarrow R_{12}$
30	XEQ 15		get loop index
31	♦LBL 17		loop through data
32	RCL IND 01		
33	ISG 01		
34	RCL 12		
35	X≠Y?		$x_i = x_j$?
36	GTO 16		
37	RCL IND 01		
38	RCL 13		
39	X≠Y?		$y_i = y_j$?
40	GTO 16		found correct pair
41	1		
42	ST− 02		
43	RCL IND 02		

STEP/LINE	KEY ENTRY	KEY CODE (67/97 only)	COMMENTS
44	STO IND 01		switch y_i and y_n
45	X<>Y		
46	ST− 02		decrease loop indices and n
47	ST− 01		
48	ST− 04		
49	RCL IND 02		switch x_i and x_n
50	STO IND 01		
51	XEQ 00		set up loop index
52	RCL 04		
53	RTN		
54	♦LBL 16		loop for next (x,y)
55	ISG 01		
56	GTO 17		data pair not found
57	0		
58	1/X		
59	♦LBL b		m'>m
60	STO 03		
61	GTO 02		
62	♦LBL B		m→a_i's
63	STO 03		
64	0		begin a series of complicated nested loops that find the best fit coefficients
65	STO 05		
66	STO 11		
67	1		
68	STO 24		
69	GTO 06		
70	♦LBL 02		
71	RCL 08		
72	RCL 09		
73	/		
74	CHS		
75	STO 11		
76	♦LBL 03		
77	RCL 06		
78	RCL 08		
79	/		
80	CHS		
81	STO 10		
82	RCL 08		
83	STO 09		
84	1		
85	ST+ 05		
86	RCL 05		
87	30		
88	+		
89	0		

Note: Refer to "HP-41C OWNER'S HANDBOOK AND PROGRAMMING GUIDE" for specific information on keystrokes. The Function Index is found at the very back of the Handbook. Refer to Appendix E i 67 or 97 "OWNER'S HANDBOOK AND PROGRAMMING GUIDE" for exact keystrokes.

PROGRAM LISTING

☐ 67 ☐ 97 ☒ 41C

STEP/LINE	KEY ENTRY	KEY CODE (67/97 only)	COMMENTS

```
 90 STO IND
Y
 91 6
 92 ST- Z
 93 RDN
 94 STO IND
Y
 95 RCL 05
 96 XEQ 14
 97*LBL 04
 98 RCL 01
 99 23
100 +
101 RCL IND
X
102 STO 13
103 RCL 10
104 *
105 7
106 ST+ Z
107 RDN
108 RCL IND
Y
109 RCL 11
110 *
111 +
112 RCL 13
113 STO IND
Z
114 RDN
115 7
116 ST- Z
117 X<>Y
118 STO IND
Z
119 ISG 01
120 GTO 04
121 RCL 05
122 XEQ 14
123*LBL 05
124 RCL 01
125 30
126 +
127 RCL IND
X
128 6
129 ST- Z
130 RDN
131 ST+ IND
Y
```

nested computations for the orthogonal polynomial fitting is too complicated to meaningfully document

STEP/LINE	KEY ENTRY	KEY CODE (67/97 only)	COMMENTS

```
132 ISG 01
133 GTO 05
134*LBL 06
135 0
136 STO 06
137 STO 07
138 STO 08
139 XEQ 15
140*LBL 11
141 1
142 STO 12
143 RCL 05
144*LBL 08
145 X=0?
146 GTO 10
147 23
148 +
149 RCL IND
X
150 RCL IND
01
151 RCL 12
152 *
153 +
154 STO 12
155 RDN
156 24
157 -
158 GTO 08
159*LBL 10
160 RCL IND
01
161 RCL 12
162 X↑2
163 ST+ 08
164 *
165 ST+ 06
166 1
167 ST+ 01
168 RCL IND
01
169 RCL 12
170 *
171 ST+ 07
172 ISG 01
173 GTO 11
174 RCL 07
175 RCL 08
176 /
```

continue

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PROGRAM LISTING

☐ 67 ☐ 97 ☒ 41C

STEP/LINE	KEY ENTRY	KEY CODE (67/97 only)	COMMENTS
177	RCL 05		
178	17		
179	+		
180	X<>Y		
181	STO IND Y		
182	STO 13		
183	RCL 05		
184	X=0?		
185	GTO 03		
186	XEQ 14		
187	♦LBL 13		
188	RCL 01		
189	23		
190	+		
191	RCL IND X		
192	RCL 13		
193	*		
194	7		
195	ST- Z		
196	RDN		
197	ST+ IND Y		
198	ISG 01		
199	GTO 13		
200	RCL 05		
201	RCL 03		
202	X>Y?		
203	GTO 02		
204	♦LBL ~~d~~ c		list the
205	RCL 03		coefficients
206	1		
207	+		compute the
208	XEQ 14		loop index
209	TONE 5		
210	♦LBL 09		
211	RCL 01		
212	FIX 0		
213	1		assemble a
214	-		label for the
215	INT		coefficient
216	"A"		
217	ARCL X		
218	"⊢="		
219	17		
220	+		
221	RCL IND X		

STEP/LINE	KEY ENTRY	KEY CODE (67/97 only)	COMMENTS
222	SCI 4		
223	ARCL X		
224	AVIEW		display the
225	FC? 21		coefficient
226	STOP		
227	ISG 01		
228	GTO 09		
229	STOP		
230	♦LBL C		compute r^2
231	0		and YMD
232	STO 16		initialize
233	STO 11		registers
234	STO 12		
235	STO 13		
236	XEQ 15		
237	♦LBL 07		loop to
238	1		find $\bar{y}$
239	ST+ 01		
240	RCL IND 01		
241	ST+ 11		$\bar{y} \to R_{11}$
242	ISG 01		
243	GTO 07		
244	RCL 04		
245	ST/ 11		
246	XEQ 15		
247	♦LBL 12		loop to find
248	RCL IND 01		sums for
249	XEQ D		r^2 and YMD
250	STO 10		$\hat{y}_i \to R_{10}$
251	RCL 11		
252	-		
253	X↑2		$R_{13} + (\hat{y}_i - \bar{y})^2$
254	ST+ 13		
255	1		
256	ST+ 01		
257	RCL IND 01		
258	RCL 11		
259	-		
260	X↑2		
261	ST+ 12		$R_{12} + (y_i - \bar{y})^2$
262	RCL IND 01		
263	RCL 10		
264	-		
265	ABS		$R_{16} + \lvert y_i - \hat{y}_i \rvert$
266	ST+ 16		

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00609C PROGRAM LISTING

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STEP/LINE	KEY ENTRY	KEY CODE (67/97 only)	COMMENTS
267	ISG 01		
268	GTO 12		
269	RCL 13		
270	RCL 12		
271	/		
272	"R↑2="		label and display r^2
273	ARCL X		
274	TONE 5		
275	AVIEW		
276	FC? 21		
277	STOP		
278	RCL 16		
279	RCL 04		
280	/		
281	"YMD="		label and display YMD
282	ARCL X		
283	AVIEW		
284	RTN		
285	STOP		
286	♦LBL 14		compute loop indices
287	1 E3		
288	/		
289	1		
290	+		
291	STO 01		
292	RTN		
293	♦LBL 00		compute index for looping through the data and save it in R_{15}
294	RCL 04		
295	2		
296	*		
297	36		
298	+		
299	1 E3		
300	/		
301	37		
302	+		
303	STO 15		
304	RTN		
305	♦LBL 15		reset the loop index
306	RCL 15		
307	STO 01		
308	RTN		
309	♦LBL D		given x, compute $\hat{y}$ from the fitted curve
310	STO 00		
311	0		
312	STO 07		
313	RCL 03		
314	STO 14		
315	♦LBL 01		
316	RCL 14		
317	17		
318	+		
319	RCL IND X		assemble the coefficients to compute the curve
320	ST+ 07		
321	RCL 00		
322	ST* 07		
323	DSE 14		
324	GTO 01		
325	RCL 07		
326	RCL 17		
327	+		
328	RTN		
329	♦LBL "L"		list all the (x,y) pairs
330	XEQ 15		
331	CLX		
332	♦LBL 18		
333	FIX 0		
334	"X"		assemble label
335	1		
336	+		
337	ARCL X		
338	"⊢="		
339	RCL IND 01		
340	SCI 4		
341	ARCL X		display x_i
342	AVIEW		
343	FC? 21		
344	STOP		
345	RDN		
346	ISG 01		
347	FIX 0		
348	"Y"		assemble label
349	ARCL X		
350	"⊢="		
351	RCL IND 01		
352	SCI 4		
353	ARCL X		
354	AVIEW		display y_i
355	FC? 21		
356	STOP		
357	RDN		
358	ISG 01		
359	GTO 18		
360	RTN		
361	END		

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00609C

REGISTERS, STATUS, FLAGS, ASSIGNMENTS

DATA REGISTERS

00	used		
	loop index		
	counter for input		
	m		
05	n		
	sum1		
	sum2		
	sum3		
	sum4		
10	Bj		
	Cj		
	used		
	used		
	used		
15	used		
	used		
	a0		
	a1		
	a2		
20	a3		
	a4		
	a5		
	a6		
	used		
	used		
	:		
35	used		
	used		
	x1 (will start	at	19+3M,
	y1 see discus	sion	in
	x2 program de	scri	ption I)
40	y2		
	:		
	xn		
	yn		

STATUS

SIZE 37+2n TOT. REG. 119+2n USER MODE

ENG ___ FIX ___ SCI 4 ON X OFF ___

DEG X RAD ___ GRAD ___

FLAGS

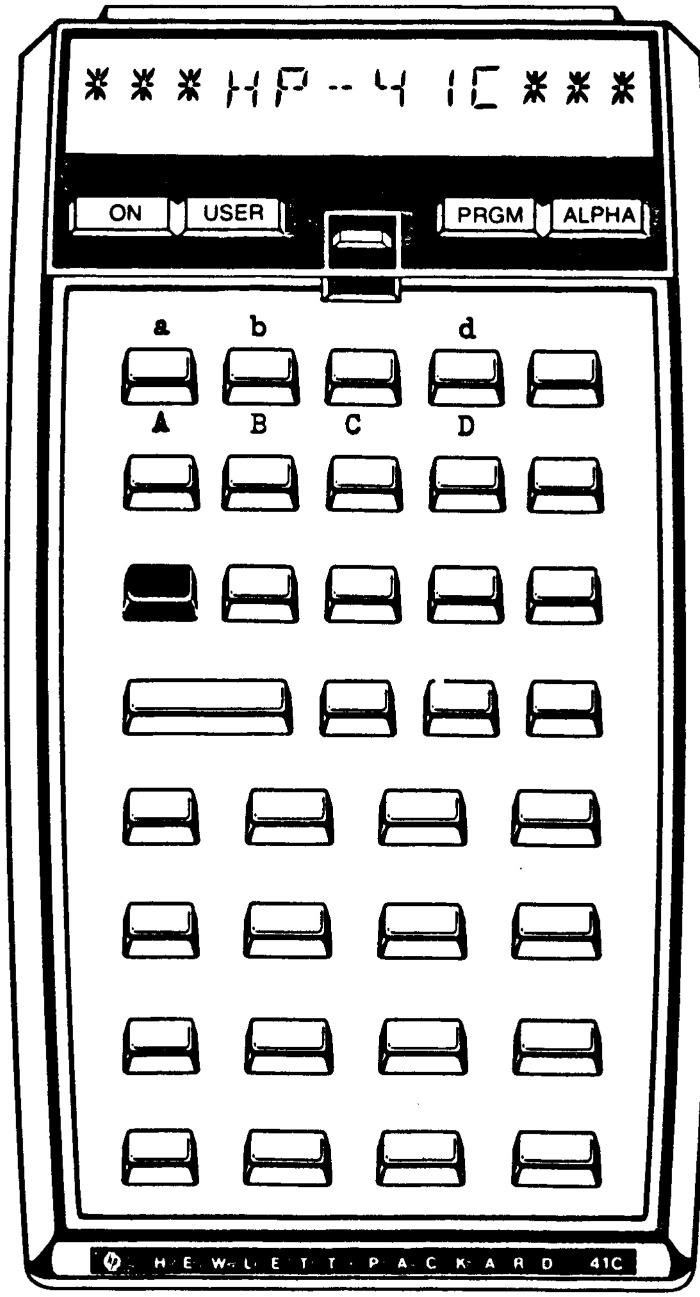
#	INIT S/C	SET INDICATES	CLEAR INDICATES
29	c		better appearing output

ASSIGNMENTS

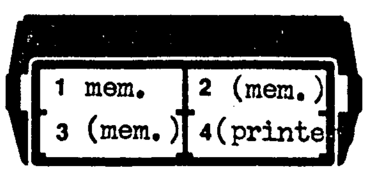
FUNCTION	KEY	FUNCTION	KEY

KEYBOARD CARD LABELING

KEYBOARD

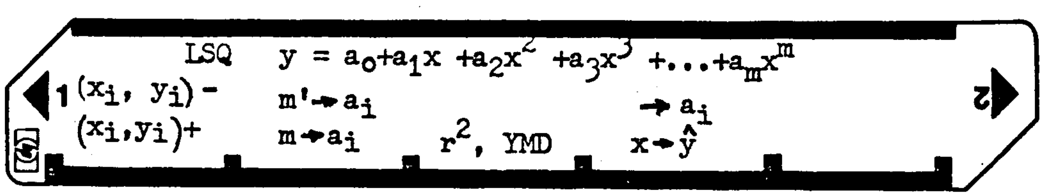


SYSTEM CONFIGURATION



1 memory module required,
more may be necessary.
printer is optional

CARD



ROW 1 (1 : 4)

ROW 2 (5 : 7)

ROW 3 (8 : 15)

ROW 4 (16 : 25)

ROW 5 (25 : 32)

ROW 6 (32 : 40)

ROW 7 (40 : 47)

ROW 8 (47 : 54)

ROW 9 (54 : 61)

ROW 10 (62 : 71)

ROW 11 (72 : 84)

ROW 12 (85 : 93)

ROW 13 (94 : 101)

ROW 14 (102 : 112)

ROW 15 (113 : 120)

ROW 16 (121 : 129)

ROW 17 (129 : 138)

ROW 18 (139 : 147)

ROW 19 (148 : 157)

ROW 20 (158 : 166)

ROW 21 (167 : 174)

ROW 22 (175 : 185)

ROW 23 (185 : 193)

ROW 24 (194 : 202)

ROW 25 (203 : 210)

ROW 26 (211 : 218)

ROW 27 (219 : 226)

ROW 28 (227 : 235)

ROW 29 (236 : 242)

ROW 30 (243 : 249)

ROW 31 (249 : 258)

ROW 32 (259 : 267)

ROW 33 (268 : 274)

ROW 34 (274 : 281)

ROW 35 (281 : 290)

ROW 36 (291 : 300)

ROW 37 (301 : 310)

ROW 38 (311 : 320)

ROW 39 (321 : 329)

ROW 40 (329 : 334)

ROW 41 (334 : 341)

ROW 42 (341 : 349)

ROW 43 (349 : 355)

ROW 44 (356 : 361)