EUROPE 50520

Program Description I

Program Title: Approximation by Legendre Polynomials up to degree 7.

Contributor's Name: Dr. Henrique E. Adler

Address: Rua de S.João de Brito, 136

City: Oporto Country: Portugal Postal Code:

Program Description, Equations, Variables A set of m+1 equally spaced data points (x_i, y_i) $(i = 0 \ldots m)$ is approximated by a sum of Legendre Polynomials $\hat{y} = \sum_0^n a_k P_k(t)$, where $n \leq 7$; $k = 0 \ldots n$; $t = px + q$; $p = 2/(x_m - x_o)$; $q = -(x_m + x_o)/(x_m - x_o)$; and $a_k = [(2k+1)/2] \int_{-1}^{+1} P_k(t) y\,dt$. $P_k(t)$ is the Legendre Polynomial of degree k.

The Legendre Polynomials of lowest dregree are:

$P_0(t) = 1$; $P_1(t) = t$; $P_2(t) = (3t^2 - 1)/2$; $P_3(t) = (5t^3 - 3t)/2$.

The recurrency relation used in this program is:

$(k+1)P_{k+1}(t) = (2k+1)tP_k(t) - kP_{k-1}(t)$.

The program computes the half sum of squared errors $M_k/2 = \frac{1}{2}\int_{-1}^{+1} y^2 dt - \sum a_k^2/(2k+1)$.

This is used for finding the best n value.

There is a choice of 3 integration formulas: Trapezium, Simpson and Newton-Cotes(5). Use Trapezium only for scattered and numerous data points. For points forming a smooth curve, use Simpson when m is even, or Newton-Cotes (best accuracy) when m is multiple of 4. The final result reads:

$\hat{y} = a_0 + a_1 t + a_2 P_2(t) + a_3 P_3(t) + \ldots + a_n P_n(t)$. If you prefer the result in form of a power series with argument x, $(\hat{y} = c_0 + c_1 x + c_2 x^2 + \ldots c_n x^n)$, convert a_i into c_i using the coefficient conversion program existing in the Users' Library. 50813 D In this case, all register contents must be conserved without any alteration.

Operating Limits and Warnings Make y entries with care. False entries cannot be corrected. The more data points you have, the better the accuracy will be. High degree approximation with few data points is impossible. After completion of step 8 of the User Instructions, pressing of R/S will cause ERROR-display, which can be cleared by pressing CLX followed by the right key.

Sketch(es)

Sample Problem(s) Approximate the following set of data points by a polynomial:

x	0	1	2	3	4	5	6	7	8	9	10
y	0	0.2	0.8	1.5	2.7	3.7	5.2	6.7	8.3	9.9	11.5

$m = 10$ (even), smooth curve: Simpson rule.

Keystrokes: 10 [↑] 0 [↑] 10 [f] [b] ⟶ 7.000

5 [R/S] (because n is certainly not greater than 5) ⟶ 0.

0 [R/S] .2 [R/S] .8 [R/S] 1.5 [R/S] 2.7 [R/S] 3.7 [R/S] 5.2 [R/S] 6.7 [R/S] 8.3 [R/S] 9.9 [R/S]

11.5 [R/S] ⟶ 5.000. (end)

	k	a_k	$M_k/2$	
[B] ⟶	k = 0.×××	a_k = 4.450×××	$M_k/2$ = 12.714×××	
	1.×××	6.086×××	0.367×××	
	2.×××	1.322×××	0.018×××	
	3.×××	−0.315×××	0.004×××	best fit with $n_1 = 3$.
	4.×××	0.051×××	0.003×××	practically no improvement
	5.×××	0.321×××	−0.006×××	with $n_1 = 4$.

Disregard a_4 and a_5. 5.000 (end)

Solution(s) $y = 4.450 + 6.086t + 1.322P_2(t) - 0.315P_3(t)$. $t = 0.2x - 1$

Prepare projections: 3 [D] ⟶ 3.000. Make projections: x [E] ⟶ $\hat{y}$

Result:

x	0	1	2	3	4	5	6	7	8	9	10
$\hat{y}$	0.00	0.21	0.74	1.53	2.56	3.79	5.17	6.68	8.27	9.90	11.54

The approximation is satisfactory.

Reference(s)

Willers, Methoden der praktischen Analysis, Berlin, 1928, p.257 ff.

Willers, Numerische Integration, Goeschen, Berlin, 1923, p.45 ff.

Meyer zur Capellen, Instrumentelle Mathematik fuer den Ingenieur, Girardet, Essen, 1952, p.371 ff.

The above sample problem was taken from Willers, Numerische Integration, p.54.

$x_m \uparrow x_o \uparrow m$ Legendre Approximation

1					2
TRAPEZE	SIMPSON	N.-C.			
	$\rightarrow k, a_k, M_k/2$	$k \rightarrow a_k$	n_1	$x \rightarrow \hat{y}$	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS		OUTPUT DATA/UNITS
1	Load side 1 and side 2.				0.000
2	Input upper limit x_m.	x_m	ENT↑		x_m
3	Input lower limit x_o.	x_o	ENT↑		x_o
4	Input m.	m			m
5	Select integration formula.				
	for Trapezium formula:		f	a	7.000
	for Simpson formula:		f	b	7.000
	for Newton-Cotes (5) formula:		f	c	7.000
6	Select degree n of polynomial.				
6a	for $n = 7$ or		R/S		0.
6b	for $n < 7$ (may be skipped)	n	R/S		0.
7	Input y value	y_i	R/S		$i+1$
8	Repeat step 7 for all y data. After the				
	last input (y_m), n is displayed.				n
9	Output coefficients a_k and sum of				
	squared errors $M_k/2$.		B		$k, a_k, M_k/2$ *)
10	Optional: Display a_k $(0 \le k \le n)$	k	C		a_k
11	Optional: Make projection based on a				
	known x value, degree n. (may be skipped)	x	E		$\hat{y}$
12	Optional: Prepare projection based on				
	a known x value, degree n_1 $(0 \le n_1 \le m)$.	n_1	D		n_1
13	Optional: Repeat step 11 for degree n_1.	x	E		$\hat{y}$

*) Last display: n. Write down all k, a_k and $M_k/2$ values. The best n value is the least k for which M_k is nearly 0. Call it n_1. Disregard all coefficients with $k > n_1$. Make projections degree n_1 (steps 12 and 13).

STEP	KEY ENTRY	KEY CODE	COMMENTS
001	*LBL a	32 25 11	Setting for Trapeze formula $b = 2$
	CF0	35 61 00	
	CF1	35 61 01	
	2	02	
	GTO 9	22 09	
	*LBL b	32 25 12	Setting for Simpson formula $b = 3$
	SF0	35 51 00	
	CF1	35 61 01	
	3	03	
010	GTO 9	22 09	
	*LBL c	32 25 13	Setting for Newton-Cotes (5) formula (N-C) $b = 22.5$
	CF0	35 61 00	
	SF1	35 51 01	
	2	02	
	2	02	
	•	83	
	5	05	
	*LBL 9	31 25 09	
	CF2	35 61 02	
020	CLREG	31 43	
	P⇄S	31 42	
	CLREG	31 43	
	X⇄Y	35 52	
	STO 3	33 03	
	X	71	
	STO 2	33 02	
	2	02	m even? (Simpson) m multiple of 4? (N-C) if not, Error.
	÷	81	
	FRAC	32 83	
030	X≠0	31 61	
	GTO A	22 11	
	R↓	35 53	
	−	51	$1/p = \frac{x_m - x_0}{2}$
	2	02	
	÷	81	
	STO C	33 13	
	−	51	$q = -\frac{x_m + x_0}{x_m - x_0}$
	RCL C	34 13	
	÷	81	
040	CHS	42	
	STO D	33 14	
	7	07	
	F1?	35 71 01	
	STO÷2	33 81 02	
	R/S	84	Store desired n ($n \leq 7$)
	STO E	33 15	
	0	00	
	*LBL 8	31 25 08	
	GSB 6	31 22 06	
050	*LBL 7	31 25 07	$\frac{y_i}{bm}$
	RCL 2	34 02	
	÷	81	
	STO 4	33 04	
	P⇄S	31 42	
	STO+0	33 61 00	
	P⇄S	31 42	

STEP	KEY ENTRY	KEY CODE	COMMENTS
	X	71	$\frac{y_i^2}{bm}$
	STO+9	33 61 09	
	RCL 5	34 05	
060	1	01	
	STO I	35 33	
	GTO 9	22 09	
	*LBL 5	31 25 05	
	GSB 3	31 22 03	
	*LBL 9	31 25 09	
	STO 0	33 00	
	R↓	35 53	
	STO 1	33 01	
	RCL 4	34 04	
070	X	71	
	P⇄S	31 42	
	STO+(i)	33 61 24	
	RCL I	35 34	
	X	71	$\frac{(2k+1)y_i}{bm}$
	STO+(i)	33 61 24	
	STO+(i)	33 61 24	
	P⇄S	31 42	
	RCL E	34 15	if $k \leq n$, start a new loop
	ISZ I	31 34	
080	RCL I	35 34	
	X≤Y	32 71	
	GTO 5	22 05	
	RCL 3	34 03	
	RCL 8	34 08	
	X=Y	32 51	
	GTO d	22 31 14	
	1	01	
	+	61	
	STO 8	33 08	
090	X=Y	32 51	if $m = k$, start last loop
	GTO 8	22 08	
	GSB 6	31 22 06	
	ENTER↑	41	
	+	61	
	F1?	35 71 01	
	GTO 0	22 00	multiply y_i by Trapeze, Simpson or N-C coefficients
	F0?	35 71 00	
	F2?	35 71 02	
	GTO 7	22 07	
100	ENTER↑	41	
	+	61	
	SF2	35 51 02	
	GTO 7	22 07	
	*LBL 0	31 25 00	
	F0?	35 71 00	
	GTO 9	22 09	
	SF0	35 51 00	
	1	01	
	6	06	
110	GTO 1	22 01	
	*LBL 9	31 25 09	
	CF0	35 61 00	

REGISTERS

0	1	2	3	4	5	6	7	8	9
$P_{k-1}(t)$	$P_k(t)$	bm	m	$\frac{y_i}{bm}$, $\hat{y}$	t, used			counter	$\frac{1}{2}\int_{-1}^{+1} y^2 dt$

S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	0	0

A	B	C	D	E	I
		1/p	q	n	k

STEP	KEY ENTRY	KEY CODE	COMMENTS
	F2?	35 71 02	
	GTO 7	22 07	
	SF2	35 51 02	
	6	06	
	*LBL1	31 25 01	
	×	71	
	7	07	
120	÷	81	
	GTO 7	22 07	
	*LBL3	31 25 03	$P_{k+1}(t) =$
	RCL5	34 05	$\left(1-\frac{1}{k+1}\right)(tP_k - P_{k-1})$
	RCL1	34 01	$+tP_k$
	×	71	
	ENT↑	41	
	ENT↑	41	
	RCL0	34 00	
	−	51	
130	1	01	
	RCLI	35 34	
	1/x	35 62	
	−	51	
	×	71	
	+	61	
	RCL1	34 01	
	RTN	35 22	
	*LBL6	31 25 06	
	DSP0	23 00	
140	R/S	84	Input y_i
	DSP3	23 03	
	RCL8	34 08	
	2	02	
	RCL3	34 03	$t = \frac{2i}{m} - 1$
	÷	81	
	×	71	
	1	01	
	−	51	
	STO5	33 05	
150	R↓	35 53	
	RTN	35 22	
	*LBLB	31 25 12	
	RCL9	34 09	
	STO5	33 05	
	0	00	
	STOI	35 33	$\frac{1}{2} M_k = \frac{1}{2}\int_{-1}^{+1} y^2 dt$
	*LBL2	31 25 02	
	DSP0	23 00	
	−x−	31 84	$-\sum_0^k \frac{a_k^2}{2k+1}$
160	ENT↑	41	
	+	61	
	1	01	
	+	61	
	GSB1	31 22 01	
	−x−	31 84	
	x^2	32 54	
	x⇄y	35 52	
	÷	81	

STEP	KEY ENTRY	KEY CODE	COMMENTS
	STO−5	33 51 05	
170	RCL5	34 05	
	−x−	31 84	
	RCLE	34 15	
	ISZI	31 34	
	RCLI	35 34	
	x≤y	32 71	
	GTO2	22 02	
	*LBLd	32 25 14	
	RCLE	34 15	
	GTO0	22 00	
180	*LBLE	31 25 15	
	P⇄S	31 42	
	RCL0	34 00	
	P⇄S	31 42	
	STO4	33 04	
	R↓	35 53	
	RCLC	34 13	
	÷	81	$t = px + q$
	RCLD	34 14	
	+	61	
190	STO5	33 05	
	1	01	
	STOI	35 33	
	GTO9	22 09	$\hat{y} = \sum_0^n a_k P_k(t)$
	*LBL4	31 25 04	
	GSB3	31 22 03	
	*LBL9	31 25 09	
	STO0	33 00	
	R↓	35 53	
	STO1	33 01	
200	GSB1	31 22 01	
	×	71	
	STO+4	33 61 04	
	RCLI	35 34	
	RCLE	34 15	
	ISZI	31 34	
	x>y	32 81	
	GTO4	22 04	
	DSP2	23 02	
	RCL4	34 04	
210	GTO0	22 00	
	*LBLC	31 25 13	
	STOI	35 33	
	*LBL1	31 25 01	
	DSP3	23 03	
	P⇄S	31 42	
	RCL(i)	34 24	
	P⇄S	31 42	
	GTO0	22 00	
	*LBLD	31 25 14	
220	STOE	33 15	
	*LBL0	31 25 00	
	RTN	35 22	
	GTO A	22 11	Lock

LABELS

A Error	B →k, a_k, $\Sigma\delta^2$	C k→a_k	D n	E x→$\hat{y}$
a TRPZ	b SMPSN	c NWTN-CTS	d	e
0	1	2	3	4
5	6	7	8	9

FLAGS

0	SMPSN
1	NWTN-CTS
2	
3	

SET STATUS

FLAGS	ON	OFF	TRIG		DISP	
0	☐	☒	DEG	☒	FIX	☒
1	☐	☒	GRAD	☐	SCI	☐
2	☐	☒	RAD	☐	ENG	☐
3	☐	☒			n	3