

Program Description I

Program Title Summation of an infinite alternating series

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Program Description, Equations, Variables This program computes the sum of alternating series of the form : $y(0) - y(1) + y(2) - y(3) + \dots = \sum_{i=0}^{\infty} (-1)^i y(i)$ where $i = 0, 1, 2, \dots$ up to infinity. This is mostly useful when the series converges rather slowly to its limit. For instance, consider the series $S = 1 - 1/2 + 1/3 - 1/4 + \dots$, whose limit is $\ln 2$. Were you to find an approximation by actually adding the series terms, you would have to add hundreds of thousands of terms to achieve a mere 4 places approximation.

Of course, there exist more efficient methods to deal with the problem. This program includes 2, the well-known Euler's transformation and the Hutton's transformation:

(1) EULER'S TRANSFORMATION .-

Consider the series

The Euler's transformation replaces $y(0) - y(1) + y(2) - y(3) + \dots$ by : $= 1/2 y(0) - 1/4 \Delta y(0) + 1/8 \Delta^2 y(0) - \dots$, where the $\Delta^n y(0)$ are the n-th order forward difference of $y(i)$; up to 17th-order differences may be used.

The program proceeds as follows: given $y(i)$, (which is defined by the user under LBL E, using 41 steps at most, and (TURN PAGE)

- Operating Limits and Warnings - increasing the number of differences and/or previous terms added increases both accuracy and running time.
- some divergent series may be treated too, but the result is not guaranteed in all cases.
 - The Euler's transf. is most efficient for very slowly convergent series. The Hutton's one deals equally well with any series.
 - $y(i)$ must be strictly positive

This program has been verified only with respect to the numerical example given in Program Description II. User accepts and uses this program material AT HIS OWN RISK, in reliance solely upon his own inspection of the program material and without reliance upon any representation or description concerning the program material.

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using R_A , R_B , if required), n terms of the series (n is a user's specified value) are summed in advance, to give S' . Then a difference table is formed thus:

$y(n+1)$				
$y(n+2)$	$\Delta y(n+1)$	$\Delta^2 y(n+1)$	$\Delta^3 y(n+1)$	
$y(n+3)$	$\Delta y(n+2)$			
.....				

computing differences up to the m -th order (m is a user's specified value, $1 \leq m \leq 17$). And finally, the Euler's transformation is applied to yield $S'' = \frac{1}{2}y(n+1) - \frac{1}{4}\Delta y(n+1) + \frac{1}{8}\Delta^2 y(n+1) - \dots$ and then, $S = S' + S''$ is the final result. Some useful remarks:

- a) The ET is most efficient for series whose convergence is very slow, so that $y(i)$ tends to zero like $1/i$ or so.
- b) Tabulation of differences may require a long running time if m is great. In general, $m \leq 10$ is advisable. I think that $n = m = 8$ is a very good election.

2) HUTTON'S TRANSFORMATION .- The Hutton's transformation deals directly with the sequence of partial sums S_k , such that:

$S_k = y(0) - y(1) + y(2) - \dots + (-1)^k y(k)$, and replaces the sequence $S_0, S_1, S_2, \dots$ by a new sequence $T_0, T_1, T_2, \dots$ where $T_k = \frac{1}{2}(S_k + S_{k-1})$, after which the transformation may be iterated. In fact, the programs sums n terms of the series in advance and then computes and stores $S_{n+1}, S_{n+2}, \dots, S_{n+9}$, after which 9 iterations of Hutton's transformation are performed. Remarks:

- a) Hutton's t. applies equally well to either slow or fast convergent series. number of terms to be summed in advance is user's defined; $n = 8$ is a good value. Accuracy & running time depends on n .
- b) The successive approximations afforded by every iteration of Hutton's t. may be printed at will, to see how the method converges. This is stated using the toggle device LBL d.

Program Description II

~~*****~~ c) Some considerations: divergent series may be treated as well. If a sum for the divergent series is obtained, it is its Euler's sum (or its Hutton's sum; if both exist, they are equal)

d) A great deal of program instructions have been duplicated, to increase program speed. Also, the iterations of Hutton's t. are straightforward, without branches, to avoid a loop within a loop.

e) Any alternating series may be summed: simply define the general term so that includes all terms of your series. See examples.

Sample Problem(s) (1) Sum the series $S = 1 - 1/2 + 1/3 - 1/4 + \dots$ to 10 places.

- define $y(i) = 1/(1+i)$: **GTO E**, switch to W/PRGM, 1 **+** **1/X** **RTN**,

switch to RUN

- now, SUM = 10, DIF = 7: 10 **B** 7 **C** **A** $\rightarrow$ 0.693147182

- Thus, the Euler's method, after 69 seconds of furious calculation gives the value 0.693147182. The exact value is $\ln 2 = 0.693147181$

- Let's see how does the Hutton's: **D** $\rightarrow$ 0.6931471805

so the Hutton's method (after 56 seconds) gives 0.6931471805, where $\ln 2 = 0.6931471806$ is the exact result.

$$(2) \text{ FIND THE VALUE OF } S = \frac{1}{2} \int_{k=0}^{k=1} \frac{\theta=\pi/2}{\theta=0} \frac{d\theta dk}{\sqrt{1-k^2 \sin^2 \theta}} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \dots$$

- DEFINE $y(i) \Rightarrow 1 - \frac{1}{3^2} + \frac{1}{5^2} - \dots = \sum_{i=0}^{\infty} \frac{1}{(2i+1)^2} \Rightarrow$ **GTO E**, SWITCH, 2 **X** 1 **+** **1/X** **X²** **RTN**, SWITCH TO RUN

Solution(s) - LET SUM = 8, DIF = 7 $\Rightarrow$ 8 **B** 7 **C** **A** $\rightarrow$ 0.915965595 (70 SECONDS)

TO TEST THE HUTTON'S METHOD **D** $\rightarrow$ 0.915965595 (58 SECONDS)

REMARKS EULER'S T. & HUTTON'S T. ARE INDEPENDENT; IF BOTH GIVE THE SAME RESULT, IT'S ALMOST SURE THAT THE RESULT IS CORRECT. IN THE LAST EXAMPLE, EXACT VALUE IS

$$S = \int_0^1 \frac{\tan^{-1} x}{x} dx = 0.915965594$$

(MORE EXAMPLES IN PAGE 4)

Reference(s) - Introduction to numerical analysis - F.B. Hildebrand
International series in pure and applied mathematics - McGraw/Hill
- Chapter 5.9 :- Approximate summation, pages 203 thru 208

(3) FIND THE SUM $S = -\frac{1}{0.23} + \frac{1}{0.24} - \frac{1}{0.25} + \frac{1}{0.26} - \dots$ TO 10 PLACES

- DEFINE GENERAL TERM $\Rightarrow S = -\left(\frac{1}{0.23} - \frac{1}{0.24} + \dots\right) = -\sum_{i=0.23}^{\infty} \pm\left(\frac{1}{i}\right) = -\sum_{i=0}^{\infty} (-1)^i \frac{1}{0.23+0.01i}$ $\Rightarrow$
 $i=0.23, 0.24, \dots$

$\Rightarrow$ [GTO E], SWITCH, [RCL A] [X] [RCL B] [+], [1/X] [RTN], SWITCH TO RUN

- STORE CONSTANTS $\Rightarrow .01$ [STOA] .23 [STOB] 6 [C] [B] [A] $\rightarrow$ 2.221127525 (55 SECONDS)

SO, EULER'S METHOD PROVIDES $S = -2.221127525$, WHERE THE EXACT

RESULT IS $S = -\int_0^1 \frac{1}{x^{0.77}(1+x^{0.01})} dx = -100 \int_0^1 \frac{t^{22}}{1+t} dt = -100 \left[\ln 2 - \sum_{i=0}^{21} (-1)^i \frac{1}{1+i} \right]$

$= -2.221127525$. TO TEST HUTTON'S METHOD $\Rightarrow$ 0 [B] $\rightarrow$ 0

[D] $\rightarrow$ 2.221127524 (41 SEC)

(4) FIND THE SUM $1 - \left(\frac{1}{3}\right)^5 + \left(\frac{1}{3}\right)^5 - \left(\frac{1}{3}\right)^5 + \dots = \sum_{i=0}^{\infty} (-1)^i (2i+1)^{-5}$

- THE PREVIOUS EXAMPLES WERE VERY SLOWLY CONVERGENT SERIES. THIS ONE CONVERGES FASTER

- DEFINE GENERAL TERM $\Rightarrow$ [GTO E], SWITCH, [2] [X] 1 [+], [RCL A] [4 X] [RTN], SWITCH TO RUN

- STORE CONSTANTS $\Rightarrow$ 5 [CHS] [STOA] 10 [B] 2 [C] [A] $\rightarrow$ 0.9961578289 (41 SECONDS)

EXACT RESULT IS $5\pi^5/1536 = 0.9961578288$, SO ET BEHAVES VERY WELL.

- LET'S SEE HOW HUTTON'S T. PERFORMS IT'S WORK. SELECT "PRINT" $\Rightarrow$ [FD] 1.000000000

4 [B] [D] $\rightarrow$ 0.996157835 $\rightarrow$ 0.996157827 $\rightarrow$ 0.996157829 $\rightarrow$ 0.996157823 $\rightarrow$
 $\rightarrow$ 0.996157829 $\rightarrow$ 0.996157829 $\rightarrow$ 0.996157830 $\rightarrow$ 0.996157829 $\rightarrow$
 $\rightarrow$ 0.996157829, [FD] $\rightarrow$ 0.000000000, "NOT PRINT"

- AS YOU MAY SEE, 3 ITERATION WOULD HAVE SUFFICE, BUT 8 WERE PERFORMED, AS THE NUMBER OF ITERATIONS IS NOT USER'S SELECTABLE.

(5) FIND THE EULER'S SUM OF $S = 1 - 1 + 1 - 1 + 1 - \dots$

- THIS SERIES IS NEITHER CONVERGENT NOR DIVERGENT, AS ITS PARTIAL SUMS ARE 0, 2,

- DEFINE GENERAL TERM $\Rightarrow$ [GTO E], SWITCH, 1 [RTN], SWITCH TO RUN

- SUM = 0, DIF = 1 $\Rightarrow$ 0 [B] 1 [C] [A] $\rightarrow$ 0.5000000000 (8 SECONDS)

THE EXACT RESULT IS $\Rightarrow \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots = y(x)$

$y(1) = 1 - 1 + 1 - 1 + \dots = \frac{1}{1+1} = \frac{1}{2} = 0.5$

- WHAT DOES HUTTON'S? $\Rightarrow$ [D] $\rightarrow$ 0.5000000000 (38 SECONDS)

(6) FIND THE EULER'S (HUTTON'S) SUM OF $1 - 2 + 3 - 4 + 5 - 6 + \dots$

- THIS IS A CLEARLY DIVERGENT SERIES

- DEFINE GENERAL TERM $\Rightarrow 1 - 2 + 3 - \dots = \sum_{i=0}^{\infty} (-1)^i (1+i)$ $\Rightarrow$ [GTO E], SWITCH, 1 [+], [RTN] IS

- SUM = 0, DIF = 1 $\Rightarrow$ 0 [B] 1 [C] [A] $\rightarrow$ 0.2500000000 (9 SECONDS)

THE EXACT RESULT IS $\Rightarrow \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \dots = y(x) \Rightarrow y(1) = 1 - 2 + 3 - \dots = \frac{1}{(1+1)^2}$

- THE HUTTON'S SUM $\Rightarrow$ [D] $\rightarrow$ 0.2500000000 (39 SECONDS)



Program Listing I

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STEP	KEY ENTRY	KEY CODE	COMMENTS	STEP	KEY ENTRY	KEY CODE	COMMENTS
001	LBL d	32 25 14	"PRINT" TOGGLE		P ≥ S	31 42	
	F ? 1	35 71 01			ISZ	31 34	
	GTO 3	22 03			1	01	
	SF 1	35 51 01	SET "PRINT"	060	STO + 8	33 61 08	ANOTHER TERM?
	1	01			RCL E	34 15	
	RTN	35 22	AUXILIAR LABEL		RCL I	35 34	
	LBL 3	31 25 03			x ≤ y	32 71	
	CF 1	35 61 01			GTO 2	22 02	YES, LOOP.
	0	00	SET "NOT PRINT"		2	02	
010	RTN	35 22			F ? 2	35 71 02	ODD?
	LBL B	31 25 12	INPUT "SUM"		CHS	42	YES, -2 IN R ₈
	STO C	33 13	STORES # OF PREV. SUM.		STO 8	33 08	
	RTN	35 22			ABS	35 64	
	LBL C	31 25 13	INPUT "DIF"	070	LBL 6	31 25 06	DIFFERENCES SCHEME
	STO E	33 15	STORES # OF DIF.			51	2 LOOPS (NESTED)
	RTN	35 22			STO I	35 33	
	LBL D	31 25 14	HUTTON'S TRANSFORMATION		STO D	33 14	
	SF 0	35 51 00			LBL 0	31 25 00	MAIN LOOP
	GTO 3	22 03			P ≥ S	31 42	
020	LBL A	31 25 11	EULER'S TRANSFORMATION		RCL (1)	34 24	COMPUTES EACH
	CF 0	35 61 00			ISZ	31 34	DIFFERENCE
	P ≥ S	31 42			STO - (1)	33 51 24	REQUIRED
	CF 2	35 61 02		080	P ≥ S	31 42	
	LBL 3	31 25 03	AUXILIAR LABEL		RCL E	34 15	
	1	01			RCL I	35 34	1ST LOOP CONTROL
	STO 8	33 08			x ≠ y	32 61	
	0	00	CLEAR S		GTO 0	22 00	
	STO 9	33 09			RCL D	34 14	
	STO I	35 33	COUNTER RESET TO 0		STO I	35 33	
30	LBL 1	31 25 01	LOOP, COMPUTES $\sum_{i=1}^n (-1)^{i-1} y(i)$		x = 0	31 51	2ND LOOP CONTROL
	GSB E	31 22 15			GTO 4	22 04	
	RCL 8	34 08			1	01	
	STO - 8	33 51 08	$y(i) \times \begin{cases} 1 \\ -1 \end{cases}$	090	GTO 6	22 06	EULER'S TRANSFORMATION
	STO - 8	33 51 08			LBL 4	31 25 04	
	x	71	COMPUTES $S' = \sum_{i=0}^{\text{SUM}} (-1)^i y(i)$		P ≥ S	31 42	
	STO + 9	33 61 09			RCL (1)	34 24	
	ISZ	31 34	ANOTHER TERM?		P ≥ S	31 42	
	RCL C	34 13	YES, LOOP		RCL 8	34 08	
	RCL I	35 34			:	81	
40	x ≤ y	32 71			STO + 9	33 61 09	$S = S' + S''$
	GTO 1	22 01			2	02	WHERE $S' = \sum_{i=0}^{\text{SUM}} (-1)^i y(i)$
	F ? 0	35 71 00	HUTTON?		CHS	42	$S'' = \frac{1}{2} y(m+1) -$
	GTO 5	22 05	YES		STO x 8	33 71 08	$-\frac{1}{4} \Delta y(m+1) + \frac{1}{8} \Delta^2 y(m)$
	STO 8	33 08	NO, EULER	100	ISZ	31 34	...
	2	02			RCL E	34 15	
	:	81	ODD OR EVEN?		RCL I	35 34	
	FRAC	32 83			x ≤ y	32 71	
	x ≠ 0	31 61			GTO 4	22 04	
	SF 2	35 51 02			GTO 9	22 09	
050	CLX	44	RESET COUNTER TO 0		LBL 5	31 25 05	OUTPUT $\sum_{i=0}^{\infty} (-1)^i y(i)$
	STO I	35 33			9	09	HUTTON'S
	LBL 2	31 25 02	LOOP, TABLE OF DIFFERENCES		STO I	35 33	
	RCL 8	34 08			LBL 7	31 25 07	PARTIAL SUMS LOOP
	GSB E	31 22 15		110	RCL C	34 13	
	P ≥ S	31 42	COMPUTES AND STORES		1	01	
	STO (1)	33 24	y(m+1), y(m+2), ...		0	00	

REGISTERS

0	1	2	3	4	5	6	7	8	9
$y(m+1)$	$\Delta y(m+1)$	$\Delta^2 y(m+1)$	$\Delta^3 y(m+1)$	$\Delta^4 y(m+1)$	$\Delta^5 y(m+1)$	$\Delta^6 y(m+1)$	$\Delta^7 y(m+1)$	$\Delta^8 y(m+1)$	$\Delta^9 y(m+1)$
S0 $\Delta^0 y(m+1)$	S1 $\Delta^1 y(m+1)$	S2 $\Delta^2 y(m+1)$	S3 $\Delta^3 y(m+1)$	S4 $\Delta^4 y(m+1)$	S5 $\Delta^5 y(m+1)$	S6 $\Delta^6 y(m+1)$	S7 $\Delta^7 y(m+1)$	S8 $\Delta^8 y(m+1)$	S9 $\Delta^9 y(m+1)$
A	B	C	D	E	F	G	H	I	J
		m = SUM	used	m = DIF				used.	

STEP	KEY ENTRY	KEY CODE	COMMENTS	STEP	KEY ENTRY	KEY CODE	COMMENTS
	+	61			RCL 9	34 09	
	RCL 1	35 34		170	P=8	31 42	$\sum_{i=0}^n (-1)^i y(i)$
	-	51	COMPUTES AND STORES		RTN	35 22	
	GSB E	31 22 15	IN RS1, RS2, --	172	LBL 3	31 25 03	"PRINT" LABEL
	RCL 8	34 08	-- RS4		1	01	
	STO- 8	33 51 08			0	00	
	STO- 8	33 51 08	THE PARTIAL SUMS		RCL 1	35 34	
	X	71			-	51	
	STO+ 9	33 61 09	$S_{m+1}, S_{m+2}, \dots, S_{m+9}$		STO 1	35 33	
	RCL 9	34 09			RCL (1)	34 24	
	P=8	31 42			-x-	31 84	PRINTS EACH SUCCESS APPROXIMATION
	STO (1)	33 24		180	LAST X	35 82	
	P=8	31 42			STO 1	35 33	
	DSZ	31 33		182	RTN	35 22	
	GTO 7	22 07	LOOP CONTROL		LBL E	31 25 15	y(i) EVALUATION
	8	08					
	STO 1	35 33					
130	P=8	31 42	HUTTON'S TRANSFORMATION				
	LBL 8	31 25 08					
	RCL 8	34 08					
	STO+ 9	33 61 09					
	2	02					
	STO: 9	33 81 09	STARTING FROM A				
	RCL 7	34 07	PREVIOUS ITERATION				
	STO+ 8	33 61 08	IT COMPUTES A NEW				
	2	02	SEQUENCE OF PARTIAL				
140	STO: 8	33 81 08	SUMS, MAKING USE				
	RCL 6	34 06	OF				
	STO+ 7	33 61 07	$T_k = \frac{1}{2}(S_k + S_{k-1})$				
	2	02					
	STO: 7	33 81 07	WHERE THE S_k				
	RCL 5	34 05	ARE STORED IN				
	STO+ 6	33 61 06	RS1 THRU RS9				
	2	02					
	STO: 6	33 81 06					
	RCL 4	34 04					
	STO+ 5	33 61 05					
150	2	02					
	STO: 5	33 81 05					
	RCL 3	34 03					
	STO+ 4	33 61 04					
	2	02					
	STO: 4	33 81 04					
	RCL 2	34 02					
	STO+ 3	33 61 03					
	2	02					
	STO: 3	33 81 03					
160	RCL 1	34 01	"PRINT"?				
	STO+ 2	33 61 02	YES				
	2	02	ITERATIONS CONTROL				
	STO: 2	33 81 02					
164	F P 1	35 71 01					
165	GSB 3	31 22 03	AUXILIAR LABEL				
	DSZ	31 33					
	GTO 8	22 08					
	LBL 9	31 25 09					

LABELS						FLAGS	SET STATUS		
FULLER	B SUM	C DIF	D HUTTON	E $i \rightarrow y(i)$	0 HUTTON ?	FLAGS		TRIG	DISP
						ON OFF			
						0	☐ ☒	DEG ☐	FIX ☒
						1	☐ ☒	GRAD ☐	SCI ☐
						2	☐ ☒	RAD ☒	ENG ☐
						3	☐ ☒		n <u>9</u>